

On the coupling of Galilean-invariant field theory to curved spacetime

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Abstract

We consider the problem of coupling Galilean-invariant quantum field theories to a fixed spacetime. We propose that to do so, one couples to Newton-Cartan geometry and in addition imposes a one-form shift symmetry. This additional symmetry imposes invariance under Galilean boosts, and its Ward identity equates particle number and momentum currents. We show that Newton-Cartan geometry subject to the shift symmetry arises in null reductions of Lorentzian manifolds, and so our proposal is realized for theories which are holographically dual to quantum gravity on Schrödinger spacetimes. We use this null reduction to efficiently form tensorial invariants under the boost and particle number symmetries. We also explore the coupling of Schrödinger-invariant field theories to spacetime, which we argue necessitates the Newton-Cartan analogue of Weyl invariance.

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1 Introduction

Consider coupling a relativistic field theory to a curved background spacetime \mathcal{M} . The reasons for doing so are manifold. The partition function of the theory on \mathcal{M} as a functional of the spacetime metric g and other background fields, $\mathcal{Z}[g; \mathcal{M}]$, efficiently encodes a host of local and non-local data about the theory. To wit, correlation functions of the stress tensor follow from the functional variation of \mathcal{Z} , and the Ward identities for the stress tensor from the invariance of the partition function under reparameterizations of coordinates. \mathcal{Z} may instead have an anomalous variation under reparameterizations, in which case one can deduce the various local and discrete anomalies from the variation. And, of course, coupling to a background spacetime prepares the way for coupling the theory to dynamical gravity, provided that it does not suffer from gravitational anomalies.

Remarkably, almost all of the things we take for granted about coupling relativistic field theory to \mathcal{M} are ill-understood when it comes to non-relativistic field theory, and in particular Galilean-invariant field theory. Part of the problem is that there are many ways to couple to \mathcal{M} if one does not have an underlying Lorentz invariance. Recall that in the relativistic setting, there is more or less a unique way of putting a theory on \mathcal{M} given special relativity and the equivalence principle. The Minkowski metric appearing in flat space field theory is just a particular example of the more general case where we endow \mathcal{M} with a (pseudo)-Riemannian metric, to which we couple the theory in such a way as to be invariant under reparameterizations of the coordinates. To our knowledge, there has yet to be a corresponding recipe for coupling Galilean-invariant field theory to \mathcal{M} . That is, there is no fully covariant prescription in terms of a geometric structure to which one couples whilst maintaining particular symmetries under which \mathcal{Z} is invariant.¹

The role of anomalous symmetries in non-relativistic field theory is rather murky for this reason. After all, one must first specify the symmetries in order to classify the potential anomalies of a field theory. But this is tantamount to deducing the correct and covariant couplings to a background spacetime and gauge fields, which is the very thing that is not understood.

In a nutshell, the particle number symmetry is the culprit responsible for this difficulty. Recall that in the Galilean algebra, there is necessarily a conserved quantity which is known either as mass or particle number. There is a corresponding symmetry which is generated by an operator M . M is a central charge in the Galilean algebra, commuting with all other generators. Unlike an ordinary conserved charge Q , however, M appears on the right-hand-side of a commutator. The bracket of momenta P_i and Galilean boosts K_j is

$$[P_i, K_j] = -i\delta_{ij}M. \quad (1.1)$$

So the particle number symmetry is intimately related to the spacetime symmetries. Now consider a Galilean-invariant field theory, which necessarily has a conserved particle number current J^μ to which we may couple a background gauge field A_μ . Imagine also

¹Two brief comments are in order. First, the situation is much better understood for non-relativistic theories without Galilean boosts, albeit only recently [1, 2]. Second, there is a significant body of work on coupling Galilean theories to spacetime. Much of that work was groundbreaking, but each element in that set suffers from at least one of the two deficiencies mentioned in the main text. See Section 2 for details.

coupling the theory to spacetime. One would reasonably expect that the commutator (1.1) rears its head in the local symmetries, via interrelations between A_μ and the rest of the spacetime geometry. In this sense, A_μ should not be an ordinary $U(1)$ connection.

Son has been progressively solving this problem, beginning with a paper with Wingate in 2005 [3] and continuing into the present [4–6]. The end result of this work is a non-relativistic notion of “general covariance,” which enumerates a list of tensors to which one couples a Galilean-invariant theory when putting it on \mathcal{M} , along with the transformation properties of these tensors under coordinate reparameterization. Recently, Son has observed [5] that these tensors constitute the defining data of Newton-Cartan geometry (see e.g. [7]). Regrettably, this “general covariance” suffers from the fact that it is not entirely covariant. In the state of the art [6], the transformation laws of all of the tensors can be formulated in a coordinate-independent way, with the exception of the transformation of the gauge field A_μ .

Nevertheless this approach is on the right track. It satisfies a number of a priori requirements, perhaps the most crucial of which is that this collection of background fields and symmetries is realized holographically. By this, we mean in the sense of holographic duality, in which certain quantum field theories are dual to quantum gravity in a higher number of dimensions. There are consistent string theory realizations of so-called Schrödinger holography [4, 8–10], in which a Galilean-invariant field theory is dual to string theory on an asymptotically Schrödinger spacetime. Already in a paper [4] that initiated Schrödinger holography, Son showed that his “general covariance” is realized in this setting.

Inspired by Son’s work, we seek to deduce the correct coupling to spacetime in a completely covariant way. Our approach is somewhat experimental: we make a proposal in Subsection 2.3, which we then subject to a number of tests. The essence of our proposal is that one should couple to the data of a Newton-Cartan structure whilst maintaining a one-form shift symmetry, which is known in the Newton-Cartan literature as invariance under Milne boosts. These boosts are absent in Son’s construction. Gauge-fixing this shift symmetry leads to Son’s formalism, as we explain in Subsection 2.7.

Perhaps the strongest check of our proposal comes in Section 3. We find that Newton-Cartan geometry and the shift symmetry automatically arise in the reduction of Lorentzian manifolds in one higher dimension along a null isometry. This is exactly the boundary geometry that appears in stringy holographic duals of Galilean-invariant field theories, and so our proposal is realized holographically.

In Section 4, we extend our proposal to account for the symmetries of scale-invariant Galilean field theories coupled to spacetime. These are the Galilean versions of conformal field theories, and the scale symmetry is specified by a dynamical critical exponent z . We remind the reader that at the particular value $z = 2$, the Galilean conformal symmetry is enhanced to the Schrödinger group. Our proposal is that Galilean CFTs are invariant under a “Weyl” rescaling of the Newton-Cartan data, wherein z encodes the relative scaling of the time and space data. Our proposal satisfies a number of checks as we describe there.

Finally in Section 5 we revisit the definition of symmetry currents and the stress tensor of the field theory, and the Ward identities obeyed by them. Our discussion strongly parallels that of [6]. These are conjugate to the Newton-Cartan data $(n_\mu, h^{\mu\nu}, v^\mu, A_\mu)$ – the

energy current is conjugate to n_μ , the spatial stress tensor to $h^{\mu\nu}$, the momentum current to v^μ , and the particle number current to A_μ . Exploiting the invariance of W under the various symmetries, we then compute the Ward identities for the one-point functions of these currents. The $U(1)$ gauge invariance implies that the number current is conserved, the shift symmetry establishes the folklore result that equates momentum and number currents,² and reparamaterization invariance computes the non-conservation of the energy current and stress tensor in terms of the other data. We also use the shift symmetry to efficiently simplify the Ward identities as in (5.29).

We conclude in Section 6. Since this article is fairly lengthy, we present a summary of our results along with a discussion of open questions that are naturally raised by our analysis. Various technical results on Newton-Cartan geometry are relegated to the Appendix.

Note: The question of coupling non-relativistic field theory to spacetime is of the general and elementary sort that is usually found in textbooks. For this reason we have indulged in a number of excursions and textbook-level exercises when the occasion suited us. The reader who is more interested in getting to the point should instead focus on Sections 2 and 3.

2 Coupling to spacetime

This Section is a composition of three major themes. The first is a review of some prerequisite material on Newton-Cartan geometry, the second a statement of our proposal for coupling Galilean-invariant theories to spacetime, and the third a sequence of sanity checks on said proposal. At the end of the Section, we make two excursions, one on Galilean-invariant Wilson lines, and another on the realization of our construction in terms of frame fields and the spin connection on the tangent bundle.

2.1 A lightning review of Newton-Cartan geometry

We begin with a discussion of Newton-Cartan geometry. Since this subject is rather foreign to the average high energy or condensed matter theorist, our review here will be self-contained. In preparing this review, we found the works [7, 12–15] to be especially helpful and recommend them to the interested reader. Throughout, we will quote the results from a number of calculations whose details may be found in Appendix A.

First things first, consider a d -dimensional, orientable manifold \mathcal{M} to which we will couple our favorite Galilean-invariant field theory. We proceed by equipping this manifold with a nowhere-vanishing one-form n_μ and a twice-contravariant symmetric tensor $h^{\mu\nu}$. The latter is semi-positive-definite with rank $d - 1$, satisfying $h^{\mu\nu}n_\nu = 0$. Roughly speaking, n_μ defines a local time direction and $h^{\mu\nu}$ gives an inverse metric on spatial slices. Together, $(\mathcal{M}, n_\mu, h^{\mu\nu})$ defines a *Galilei structure*. In virtually all of the Newton-Cartan literature, n_μ is taken to be a closed one-form, $dn = 0$. However, as emphasized

²As an aside, one can add disordered sources in a way consistent with this shift symmetry, so that the relation $\mathcal{P}^i = J^i$ can hold even in impure systems (this is in contrast with commonly and reasonably held beliefs about this equality, as found in e.g. [11]). That being said, the shift symmetry is rather delicate and we expect that it only approximately holds in real-world systems.

in [1, 2, 6], n_μ should be understood as a source which couples to the energy current of quantum field theories coupled to Newton-Cartan geometry, and so it is expedient to not restrict its derivative. In fact, restricting n to be closed may lead to a number of misleading conclusions about Newton-Cartan geometry, as we will see below.

Next, we would like to define a covariant derivative, which acts on e.g. a $(1, 1)$ tensor $\mathfrak{T}^\mu{}_\nu$ as

$$D_\mu \mathfrak{T}^\nu{}_\rho = \partial_\mu \mathfrak{T}^\nu{}_\rho + \Gamma^\nu{}_{\sigma\mu} \mathfrak{T}^\sigma{}_\rho - \Gamma^\sigma{}_{\rho\mu} \mathfrak{T}^\nu{}_\sigma. \quad (2.1)$$

In analogy with Riemannian geometry, one natural possibility would be to define a torsionless derivative under which the Galilei data $(n_\mu, h^{\mu\nu})$ is constant. This does not work for two reasons: (i.) when n_μ has a nonzero exterior derivative, $dn \neq 0$, we cannot simultaneously maintain both torsionlessness and the constancy of n_μ , and (ii.) even when $dn = 0$, the resulting derivative is only determined up to a two-form $F_{\mu\nu}$.

One criterion that leads to a unique choice of the derivative is the following. We introduce a two-form $F_{\mu\nu}$ along with a nowhere-vanishing velocity vector v^μ satisfying $v^\mu n_\mu = 1$. Together with the Galilei data, the velocity algebraically defines a twice-covariant symmetric tensor $h_{\mu\nu}$ (which we caution is not the inverse of the non-invertible tensor $h^{\mu\nu}$) satisfying

$$h_{\mu\nu} v^\nu = 0, \quad h_{\mu\rho} h^{\nu\rho} = P_\mu^\nu = \delta_\mu^\nu - v^\nu n_\mu. \quad (2.2)$$

With this data in hand, we demand that the covariant derivative keeps $(n_\mu, h^{\mu\nu})$ constant and that the torsion is purely temporal. By this, we mean that the torsion $T^\mu{}_{\nu\rho} \equiv \Gamma^\mu{}_{\nu\rho} - \Gamma^\mu{}_{\rho\nu}$ satisfies $h_{\mu\sigma} T^\sigma{}_{\nu\rho} = 0$.³ Then the derivative is still ambiguous up to a two-form, which we take to be $F_{\mu\nu}$. The end result is that the connection and its torsion are (see Appendix A.1 for details)

$$\begin{aligned} \Gamma^\mu{}_{\nu\rho} &= v^\mu \partial_\rho n_\nu + \frac{1}{2} h^{\mu\sigma} (\partial_\nu h_{\rho\sigma} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho}) + h^{\mu\sigma} n_{(\nu} F_{\rho)\sigma}, \\ T^\mu{}_{\nu\rho} &= v^\mu (\partial_\rho n_\nu - \partial_\nu n_\rho), \end{aligned} \quad (2.3)$$

where we denote (anti-)symmetrization with (square) round brackets,

$$A^{(\mu\nu)} = \frac{1}{2} (A^{\mu\nu} + A^{\nu\mu}), \quad A^{[\mu\nu]} = \frac{1}{2} (A^{\mu\nu} - A^{\nu\mu}). \quad (2.4)$$

It is easy to check that $\Gamma^\mu{}_{\nu\rho}$ transforms as a connection under coordinate reparameterizations. It also does not take too much work to derive the identity

$$F_{\mu\nu} = -2h_{\rho[\mu} D_{\nu]} v^\rho, \quad (2.5)$$

from which it follows that the *geodesic acceleration* $\dot{v}^\mu \equiv v^\nu D_\nu v^\mu$ and *curl* $D^\mu v^\nu - D^\nu v^\mu$ of the velocity are given by

$$\dot{v}^\mu = -F^\mu{}_\nu v^\nu, \quad D^\mu v^\nu - D^\nu v^\mu = F^{\mu\nu}, \quad (2.6)$$

³In this work we exclusively consider Newton-Cartan geometry with vanishing spatial torsion. However there is no technical obstruction to restoring it, as may be appropriate for the study of elastic media with dislocations.

where we have raised the indices on $F_{\mu\nu}$ and D_μ with $h^{\mu\nu}$, i.e. $D^\mu = h^{\mu\nu}D_\nu$. So the two-form ambiguity in the derivative precisely corresponds to the anti-symmetric part of the derivative of v^μ .

Before going on, we observe that the term with $F_{\mu\nu}$ in (2.3) amounts to a tensorial redefinition of the connection Γ . As a result, it is a convention to include it in the definition of the covariant derivative.⁴

As a byproduct of defining the velocity vector and so $h_{\mu\nu}$, we obtain a local expression for the volume form on \mathcal{M} . First, we define the rank d tensor and its determinant

$$\gamma_{\mu\nu} \equiv n_\mu n_\nu + h_{\mu\nu}, \quad \gamma = \det(\gamma_{\mu\nu}). \quad (2.7)$$

Then the volume form is

$$\text{vol}(\mathcal{M}) = \frac{1}{d!} \varepsilon_{\mu_1 \dots \mu_d} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_d}, \quad \varepsilon_{\mu_1 \dots \mu_d} = \sqrt{\gamma} \epsilon_{\mu_1 \dots \mu_d}, \quad (2.8)$$

where $\epsilon_{\mu_1 \dots \mu_d}$ is the fully antisymmetric tensor density with $\epsilon_{01 \dots d-1} = +1$. In simpler terms, the volume form is just $d^d x \sqrt{\gamma}$.

The curvature of the derivative is defined in the usual way, through the commutator of covariant derivatives, e.g.

$$[D_\rho, D_\sigma] \mathbf{v}^\mu = R^\mu{}_{\nu\rho\sigma} \mathbf{v}^\nu, \quad (2.9)$$

which leads to the expression

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu{}_{\nu\sigma} - \partial_\sigma \Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\alpha\rho} \Gamma^\alpha{}_{\nu\sigma} - \Gamma^\mu{}_{\alpha\sigma} \Gamma^\alpha{}_{\nu\rho}. \quad (2.10)$$

When $dn = 0$, one can further restrict the connection Γ to be *Newtonian*, which means that one demands that the curvature satisfies

$$R^{[\mu}{}_{(\nu}{}^{\rho]}{}_{\sigma)} = 0, \quad (2.11)$$

where the third index is raised with $h^{\mu\nu}$ (see e.g. [12]). In ordinary Riemannian geometry, this is a symmetry of the curvature provided that we raise the third index with the inverse Riemannian metric. However, since the underlying geometry here is not Riemannian, (2.11) is a non-trivial constraint on the connection. One can straightforwardly obtain

$$R^{[\mu}{}_{(\nu}{}^{\rho]}{}_{\sigma)} = \frac{1}{2} h^{\mu\alpha} h^{\nu\beta} n_{(\nu} (dF)_{\sigma)\alpha\beta}, \quad (dF)_{\mu\nu\rho} = \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu}, \quad (2.12)$$

where we have assumed that n is closed. Thus, when $dn = 0$, the Newtonian condition is equivalent to the constraint that F is closed, $dF = 0$, in which case it may be represented locally through a $U(1)$ connection $F = dA$. When $dn \neq 0$, we have not found a suitable generalization in the literature of what it means to have a Newtonian connection. So we

⁴In their studies of Newton-Cartan geometry and the quantum Hall effect, Son and collaborators have removed the term involving $F_{\mu\nu}$ from the connection [5, 6]. That is, they take the velocity to be both geodesic and curl-free. However, this is not the same thing as setting $F_{\mu\nu} = 0$. They allow for nonzero $F_{\mu\nu}$ and indeed $F_{\mu\nu}$ appears in various tensors which they construct. In contrast, a Newton-Cartan structure where $F_{\mu\nu}$ never appears (i.e. the geometric data is only $(n_\mu, h^{\mu\nu}, v^\mu)$) is known as a Newton-Cartan-Milne structure [14].

will make our own definition, which amounts to the choice which retains $dF = 0$. We find the condition rather cumbersome and unenlightening, and so relegate it to Appendix A.2. In Section 3 we will see that a Newton-Cartan structure with a Newtonian connection in this sense emerges from the null reduction of Lorentzian manifolds, and so is a natural definition after all.

How should we think of $F_{\mu\nu}$? We remind the reader that Galilean invariance in flat space is tied up with spacetime symmetries. Here, we find a $U(1)$ connection whose field strength is naturally twisted into the gravitational connection Γ . So it is not unreasonable that A_μ should be understood as the $U(1)$ connection which couples to the particle number current. We will soon provide evidence that this is the case.

In summary, a Newton-Cartan structure with a Newtonian connection is a quintuple $(\mathcal{M}, n_\mu, h^{\mu\nu}, v^\mu, A_\mu)$ with a covariant derivative defined through the torsionful connection (2.3). In a slight abuse of terminology, we will refer to this ensemble as a Newton-Cartan structure, and drop the reference to Newtonian connections.

2.2 Milne boosts

In order to define the covariant derivative in the previous Subsection, we introduced the velocity vector v^μ normalized such that $v^\mu n_\mu = 1$. This introduction is not unique. From v^μ and a one-form ψ_μ we could define another velocity vector $(v')^\mu$ which still satisfies $(v')^\mu n_\mu = 1$ via

$$(v')^\mu = v^\mu + h^{\mu\nu} \psi_\nu. \quad (2.13a)$$

Correspondingly, we redefine $h_{\mu\nu}$ so that the relations (2.2) continue to hold, e.g. $(h')_{\mu\nu} (v')^\nu = 0$, which fixes

$$(h')_{\mu\nu} = h_{\mu\nu} - (n_\mu P_\nu^\rho + n_\nu P_\mu^\rho) \psi_\rho + n_\mu n_\nu h^{\rho\sigma} \psi_\rho \psi_\sigma. \quad (2.13b)$$

Let us take n_μ to be closed for the moment. Then there is a unique additive redefinition of A_μ which together with (2.13a) and (2.13b) leaves the connection Γ in (2.3) invariant. It is

$$(A')_\mu = A_\mu + P_\mu^\nu \psi_\nu - \frac{1}{2} n_\mu h^{\nu\rho} \psi_\nu \psi_\rho. \quad (2.13c)$$

When n_μ is not closed, the story is slightly more complicated, as we explain below. In the Newton-Cartan literature (see e.g. [14]), the redefinitions (2.13) are known as *Milne boosts*. Note that these transformations mix the geometric data v^μ with the connection A_μ . Moreover, the Milne boosts only depend on the transverse part of ψ_μ .

Before seeing what happens to the Milne boosts when dn is nonzero, let us first make a comment about how we should regard the Milne boosts. If we couple a field theory with a $U(1)$ global symmetry to the Newton-Cartan data $(n_\mu, h^{\mu\nu}, v^\mu, A_\mu)$, we can of course do so in a way that respects coordinate reparameterizations and $U(1)$ gauge invariance, but not the Milne boosts. It is a further choice not contained in Newton-Cartan geometry to impose invariance under the boosts. This point is sometimes worded unclearly or incorrectly in the Newton-Cartan literature, as in [14, 15].

Now let us not restrict n_μ to be closed. Denoting the additive variation of an object

under the Milne boosts with a Δ_ψ , we find that the connection Γ in (2.3) varies as

$$\Delta_\psi \Gamma^\mu{}_{\nu\rho} = h^{\mu\sigma} \left\{ \left(\partial_{[\rho} n_{\nu]} P_\sigma^\alpha + \partial_{[\sigma} n_{\nu]} P_\rho^\alpha + \partial_{[\sigma} n_\rho] P_\nu^\alpha \right) \psi_\alpha + \frac{\psi^2}{2} \left(n_\nu \partial_{[\rho} n_{\sigma]} + n_\rho \partial_{[\nu} n_{\sigma]} \right) \right. \\ \left. + n_\nu \partial_{[\rho} \left(\Delta_\psi A_{\sigma]} - P_{\sigma]}^\alpha \psi_\alpha + \frac{1}{2} n_{\sigma]} \psi^2 \right) + n_\rho \partial_{[\nu} \left(\Delta_\psi A_{\sigma]} - P_{\sigma]}^\alpha \psi_\alpha + \frac{1}{2} n_{\sigma]} \psi^2 \right) \right\} \quad (2.14)$$

At $dn = 0$, we see that (2.13c) is indeed the unique redefinition of A_μ which leaves the connection invariant. However, no such redefinition exists when $dn \neq 0$. That is, the variation of Γ is

$$\Delta_\psi \Gamma^\mu{}_{\nu\rho} = h^{\mu\sigma} \left\{ \left(\partial_{[\rho} n_{\nu]} P_\sigma^\alpha + \partial_{[\sigma} n_{\nu]} P_\rho^\alpha + \partial_{[\sigma} n_\rho] P_\nu^\alpha \right) \psi_\alpha + \frac{\psi^2}{2} \left(n_\nu \partial_{[\rho} n_{\sigma]} + n_\rho \partial_{[\nu} n_{\sigma]} \right) \right\}. \quad (2.15)$$

We can ameliorate this problem by redefining Γ with terms that explicitly involve the $U(1)$ connection rather than its field strength. To be precise, we define

$$(\Gamma_A)^\mu{}_{\nu\rho} \equiv \Gamma^\mu{}_{\nu\rho} + h^{\mu\sigma} \left(-A_\sigma \partial_{[\rho} n_{\nu]} + A_\nu \partial_{[\rho} n_{\sigma]} + A_\rho \partial_{[\nu} n_{\sigma]} \right) \\ = v_A^\mu \partial_\rho n_\nu + \frac{1}{2} h^{\mu\sigma} \left(\partial_\nu (h_A)_{\rho\sigma} + \partial_\rho (h_A)_{\nu\sigma} - \partial_\sigma (h_A)_{\nu\rho} \right), \quad (2.16)$$

where in the last line we have simplified the connection by defining the Milne-invariant (but not $U(1)$ -invariant) objects

$$v_A^\mu = v^\mu - h^{\mu\nu} A_\nu, \quad (h_A)_{\mu\nu} = h_{\mu\nu} + n_\mu A_\nu + n_\nu A_\mu. \quad (2.17)$$

As an aside, these objects themselves satisfy the defining relations (2.2) via

$$(h_A)_{\mu\nu} (v_A)^\nu = 0, \quad (h_A)_{\mu\rho} h^{\nu\rho} = (P_A)_\mu^\nu = \delta_\mu^\nu - v_A^\nu n_\mu.$$

Anyway, the connection Γ_A is invariant under Milne boosts (2.13), but it has a nonzero variation under $U(1)$ gauge transformations $\delta_\Lambda A_\mu = \partial_\mu \Lambda$,

$$\delta_\Lambda (\Gamma_A)^\mu{}_{\nu\rho} = h^{\mu\sigma} \left\{ -\partial_\sigma \Lambda \partial_{[\rho} n_{\nu]} + \partial_\nu \Lambda \partial_{[\rho} n_{\sigma]} + \partial_\rho \Lambda \partial_{[\nu} n_{\sigma]} \right\}. \quad (2.18)$$

So we can choose for the covariant derivative to be invariant under either $U(1)$ -invariant or boost-invariant, but not both simultaneously.

At this stage, it may strike the reader as strange to consider a redefinition which generally changes the covariant derivative or makes the derivative non-invariant under $U(1)$ gauge transformations. Nevertheless we will provide evidence that imposing invariance under Milne boosts amounts to imposing Galilean boost invariance, and we will thereby find much fruit.

2.3 The proposal

We are now in a position to precisely state our proposal. Given a Galilean-invariant field theory, it should be coupled to a Newton-Cartan structure $(n_\mu, h^{\mu\nu}, v^\mu, A_\mu)$ in such

a way that the action is invariant under coordinate reparameterizations, $U(1)$ gauge transformations, and the Milne boosts (2.13). Correspondingly, the generating functional W of correlation functions (where we take $W = -i \ln \mathcal{Z}$ for \mathcal{Z} the partition function) is an invariant functional of the Newton-Cartan data $W = W[n_\mu, h^{\mu\nu}, v^\mu, A_\mu]$.

Later in Section 5, we will define various currents through variations of W with respect to the Newton-Cartan data. The invariance of W under reparameterizations, &c, will thereby lead to Ward identities which we compute there.

2.4 Relation to the Galilean algebra

Having made our proposal, we now perform a sequence of basic sanity checks on it. The first is to verify that the global symmetries of the flat Newton-Cartan structure on \mathbb{R}^d are generated by the Galilean algebra. The analogous statement in pseudo-Riemannian geometry is that the global symmetries (the isometries) of the flat Minkowski metric on \mathbb{R}^d are generated by the Poincaré algebra.

Consider an infinitesimal coordinate reparameterization ξ^μ , Milne boost ψ_μ , and $U(1)$ gauge transformation Λ , which we collectively notate as $\chi = (\xi^\mu, \psi_\mu, \Lambda)$. The infinitesimal variation δ_χ of the Newton-Cartan data $(n_\mu, h^{\mu\nu}, v^\mu, A_\mu)$ under the transformation χ is given by

$$\begin{aligned}\delta_\chi n_\mu &= \mathcal{L}_\xi n_\mu = \xi^\nu \partial_\nu n_\mu + n_\nu \partial_\mu \xi^\nu, \\ \delta_\chi h^{\mu\nu} &= \mathcal{L}_\xi h^{\mu\nu} = \xi^\rho \partial_\rho h^{\mu\nu} - h^{\mu\sigma} \partial_\sigma \xi^\nu - h^{\sigma\nu} \partial_\sigma \xi^\mu, \\ \delta_\chi v^\mu &= \mathcal{L}_\xi v^\mu + h^{\mu\nu} \psi_\nu = \xi^\nu \partial_\nu v^\mu - v^\nu \partial_\nu \xi^\mu + h^{\mu\nu} \psi_\nu, \\ \delta_\chi A_\mu &= \mathcal{L}_\xi A_\mu + P_\mu^\nu \psi_\nu + \partial_\mu \Lambda = \xi^\nu \partial_\nu A_\mu + A_\nu \partial_\mu \xi^\nu + P_\mu^\nu \psi_\nu + \partial_\mu \Lambda,\end{aligned}\tag{2.19}$$

where \mathcal{L}_ξ is the Lie derivative along ξ^μ . These transformations generate an algebra with $[\delta_{\chi_1}, \delta_{\chi_2}] = \delta_{\chi_{[12]}}$, where $\chi_i = (\xi_i^\mu, \psi_\mu^i, \Lambda_i)$ and $\chi_{[12]}$ is the commutator of variations, $\chi_{[12]} = (\xi_{[12]}^\mu, \psi_\mu^{[12]}, \Lambda_{[12]})$ and is given in terms of the individual variations as

$$\begin{aligned}\xi_{[12]}^\mu &= \mathcal{L}_{\xi_1} \xi_2^\mu = \xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu, \\ \psi_\mu^{[12]} &= \mathcal{L}_{\xi_1} \psi_\mu^2 - \mathcal{L}_{\xi_2} \psi_\mu^1 = \xi_1^\nu \partial_\nu \psi_\mu^2 + \psi_\nu^2 \partial_\mu \xi_1^\nu - \xi_2^\nu \partial_\nu \psi_\mu^1 - \psi_\nu^1 \partial_\mu \xi_2^\nu, \\ \Lambda_{[12]} &= \mathcal{L}_{\xi_1} \Lambda_2 - \mathcal{L}_{\xi_2} \Lambda_1 = \xi_1^\mu \partial_\mu \Lambda_2 - \xi_2^\mu \partial_\mu \Lambda_1.\end{aligned}\tag{2.20}$$

The flat Newton-Cartan structure on \mathbb{R}^d is given by⁵

$$n_\mu dx^\mu = dx^0, \quad h^{\mu\nu} \partial_\mu \otimes \partial_\nu = \delta^{ij} \partial_i \otimes \partial_j, \quad v^\mu \partial_\mu = \partial_0, \quad A = 0, \tag{2.21}$$

where we have labeled the coordinates as (x^0, x^i) for $i = 1, \dots, d-1$. The global symmetries of the flat structure are generated by those infinitesimal transformations K such that δ_K vanishes when acting on the geometric data (2.21).⁶ After some straightforward

⁵Any background with a constant $v^\mu \partial_\mu = \partial_0 + v^i \partial_i$ and $A = 0$ is related to this one by a Milne boost and $U(1)$ gauge transformation, and so it suffices to consider this parameterization of the velocity when computing the global symmetries.

⁶This definition is natural from the point of view of high-energy physics, but it seems to be uncommon in the Newton-Cartan literature. The centrally extended Galilean group was obtained in just this way in [13],

computation we find that the most general such K in $d > 1$ is a linear combination of the following generators,

$$H = (-\partial_0, 0, 0), \quad P_i = (-\partial_i, 0, 0), \quad (2.22a)$$

$$R_{ij} = (x^j \partial_i - x^i \partial_j, 0, 0), \quad K_i = (-x^0 \partial_i, -dx^i, x^i), \quad (2.22b)$$

$$M = (0, 0, 1). \quad (2.22c)$$

We compute the algebra of these generators via (2.20), from which we find

$$\begin{aligned} [R_{ij}, R_{kl}] &= \delta^{ik} R_{jl} - \delta^{il} R_{jk} + \delta^{jl} R_{ik} - \delta^{jk} R_{il}, \\ [R_{ij}, P_k] &= \delta^{ik} P_j - \delta^{jk} P_i, \quad [R_{ij}, K_k] = \delta^{ik} K_j - \delta^{jk} K_i, \\ [P_i, K_j] &= -\delta^{ij} M, \quad [H, K_i] = -P_i, \end{aligned} \quad (2.23)$$

with all other commutators vanishing. Note that M is central. This is of course the Galilean algebra expressed in terms of anti-Hermitian generators. To obtain a Hermitian basis, one could redefine all of the generators by a factor of $-i$, which would have the effect of redefining the right-hand-side of each commutator by a factor of i .

(2.22) and (2.23) are the first successes of our proposal. Before going on to our second sanity check, it is worthwhile to examine how the various parts of our proposal were required in order to get (2.22) and (2.23). First, if we did not impose invariance under Milne boosts, then it is easy to show that the global symmetries would have instead been generated by the subalgebra spanned by $\{H, P_i, R_{ij}, M\}$. Second, if we did not demand the Newtonian condition (effectively $F = dA$), then there would be no $U(1)$ connection A_μ , no invariance under $U(1)$ gauge transformations, and so no central extension M . Moreover, (2.22) and (2.23) implicitly support our identification of A_μ as the connection which couples to particle number. The generator M in (2.22), which we independently understand as the particle number charge operator, generates constant phases for quantum fields charged under the $U(1)$. So M is exactly the conserved charge for the current which couples to A_μ .

2.5 Galilean free fields

Our next sanity check is to show that the simplest Galilean-invariant theory, that of a free charged field (a scalar or fermion), can be coupled to Newton-Cartan geometry in an invariant way.⁷ Consider the free-field action

$$S_{free} = \int d^d x \left\{ \frac{i}{2} \left(\Psi^\dagger D_0 \Psi - (D_0 \Psi^\dagger) \Psi \right) + \frac{\delta^{ij}}{2m} D_i \Psi^\dagger D_j \Psi \right\}, \quad (2.24)$$

but in [15] for instance, three different notions of global symmetries in Newton-Cartan geometry are presented, none of which is the one described here. Indeed, none of those definitions is sufficiently refined to identify the centrally extended Galilean algebra as in (2.22) and (2.23).

⁷In a typical discussion of Galilean-invariant theories, fields which are charged under particle number transform under *projective* representations of the Galilean group. That is, in the change of coordinates to a boosted frame, fields acquire a spacetime-dependent phase which depends on their mass m . However, we emphasize that in our construction quantum fields transform under *linear* representations of the reparameterization/Milne/ $U(1)$ symmetries.

where Ψ couples to A_μ with charge m , i.e. its covariant derivative is given by $D_\mu \Psi = \partial_\mu \Psi - imA_\mu \Psi$. We will henceforth shorthand $\Psi^\dagger \overleftrightarrow{D}_\mu \Psi = \Psi^\dagger D_\mu \Psi - (D_\mu \Psi^\dagger) \Psi$. Note that m appears as the charge fields carry under particle number. If one has a system in which all fields carry charge m , then one can rescale the gauge field as $mA_\mu = \tilde{A}_\mu$ so that all fields have charge 1.

The natural covariant generalization of (2.24) is

$$S_{cov} = \int d^d x \sqrt{\gamma} \left\{ \frac{iv^\mu}{2} \Psi^\dagger \overleftrightarrow{D}_\mu \Psi - \frac{h^{\mu\nu}}{2m} D_\mu \Psi^\dagger D_\nu \Psi \right\}. \quad (2.25)$$

This action is obviously independent under coordinate reparameterizations and $U(1)$ gauge transformations, but what about Milne boosts? Note that although the matrix $\gamma_{\mu\nu}$ defined in (2.7) transforms under Milne boosts, its determinant does not so that $\sqrt{\gamma}$ is Milne-invariant. Next, it is instructive to rewrite (2.25) as

$$S_{cov} = \int d^d x \sqrt{\gamma} \left\{ -\frac{m}{2} (h^{\mu\nu} A_\mu A_\nu - 2v^\mu A_\mu) \Psi^\dagger \Psi + \frac{i}{2} (v^\mu - h^{\mu\nu} A_\nu) \Psi^\dagger \overleftrightarrow{\partial}_\mu \Psi - \frac{h^{\mu\nu}}{2m} \partial_\mu \Psi^\dagger \partial_\nu \Psi \right\}, \quad (2.26)$$

and recall that $v^\mu - h^{\mu\nu} A_\nu$ and $h^{\mu\nu}$ are Milne-invariant. It is easy to show that the scalar $h^{\mu\nu} A_\mu A_\nu - 2v^\mu A_\mu$ is also Milne-invariant, which shows that S_{cov} is invariant too.

Note that it is easy to add interactions to the free field theory (2.24) in a covariant way. By our analysis here, any action of the form

$$S = \int d^d x \sqrt{\gamma} \mathcal{L}(\mathcal{K}_{ij}, \Psi_i^\dagger, \Psi_j), \quad (2.27)$$

$$\mathcal{K}_{ij} \equiv \frac{iv^\mu}{2} (m_i \Psi_i^\dagger D_\mu \Psi_j - m_j (D_\mu \Psi_i^\dagger) \Psi_j) - \frac{h^{\mu\nu}}{2} D_\mu \Psi_i^\dagger D_\nu \Psi_j,$$

where Ψ_i carries charge m_i and \mathcal{L} is a $U(1)$ singlet, is automatically invariant under coordinate reparameterizations, $U(1)$ gauge transformations, and Milne boosts.

2.6 Magnetic moments and modified Milne boosts

In two spatial dimensions, Son [5] has added a magnetic moment g_s to the field theory of the previous subsection, in such a way that it is invariant under his “non-relativistic general covariance.” Very recently [6], that theory has been coupled to a more general spacetime background. This theory has a significant connection to the phenomenology of quantum Hall physics. Here we would like to understand the g_s coupling in a fully covariant way.

The action written down in [6] is

$$S_{Son} = \int dx^0 d^2 x \sqrt{g} e^{-\Phi} \left\{ \frac{ie^\Phi}{2} \Psi^\dagger \overleftrightarrow{D}_0 \Psi - \frac{1}{2m} \left(g^{ij} + \frac{ig_s}{2} \epsilon^{ij} \right) \tilde{D}_i \Psi^\dagger \tilde{D}_j \Psi \right\}, \quad (2.28)$$

where g_{ij} is a spatial metric which depends on space and time, \sqrt{g} is the square root of its determinant, and g^{ij} is its inverse. Furthermore $\tilde{D}_i = D_i + \beta_i D_0$ for β_i a vector which

depends on space and time, and ε^{ij} is a spatial epsilon tensor. It is given by $\varepsilon^{ij} = \epsilon^{ij} / \sqrt{g}$ with ϵ^{ij} the two-dimensional epsilon symbol under the convention that $\epsilon^{12} = +1$ and $\epsilon^{0i} = 0$.

There is an obvious covariant generalization of (2.28), namely

$$S_g = \int d^3x \sqrt{\gamma} \left\{ \frac{iv^\mu}{2} \varphi^* \overleftrightarrow{D}_\mu \varphi - \frac{1}{2m} \left(h^{\mu\nu} + \frac{ig_s}{2} \varepsilon^{\mu\nu} \right) D_\mu \varphi^* D_\nu \varphi \right\}, \quad (2.29)$$

where it only remains to specify what we mean by $\varepsilon^{\mu\nu}$. Recall that the volume form on \mathcal{M} is given by $\varepsilon_{\mu\nu\rho} = \sqrt{\gamma} \epsilon_{\mu\nu\rho}$ with $\epsilon_{\mu\nu\rho}$ the three-dimensional epsilon symbol. Similarly, we can define a fully antisymmetric contravariant tensor $\varepsilon^{\mu\nu\rho} = \epsilon^{\mu\nu\rho} / \sqrt{\gamma}$ with $\epsilon^{\mu\nu\rho}$ again the epsilon symbol. From this we define a spatial epsilon tensor

$$\varepsilon^{\mu\nu} = \varepsilon^{\rho\mu\nu} n_\rho = \frac{\epsilon^{\rho\mu\nu} n_\rho}{\sqrt{\gamma}}, \quad (2.30)$$

which is Milne-invariant, and this is the object which resides in the last term of (2.29).

Each term in (2.29) is manifestly invariant under coordinate reparameterizations and $U(1)$ gauge transformations. What about Milne boosts? As in the previous Subsection, it is useful to rewrite the action, this time as

$$S_g = \int d^3x \sqrt{\gamma} \left\{ -\frac{m}{2} \left(A^2 - 2v \cdot A + \frac{g_s}{2m} \varepsilon^{\mu\nu\rho} n_\mu A_\nu \partial_\rho \right) \Psi^\dagger \Psi + \frac{i}{2} (v^\mu - h^{\mu\nu} A_\nu) \Psi^\dagger \overleftrightarrow{\partial}_\mu \Psi \right. \\ \left. - \frac{h^{\mu\nu}}{2m} \partial_\mu \Psi^\dagger \partial_\nu \Psi - \frac{ig_s}{4m} \varepsilon^{\mu\nu\rho} n_\mu \partial_\nu \Psi^\dagger \partial_\rho \Psi \right\}. \quad (2.31)$$

Integrating the g_s term in the first line by parts, we see that the action S_g is Milne invariant if the objects

$$A^2 - 2v \cdot A + \frac{g_s}{2m} \varepsilon^{\mu\nu\rho} \partial_\mu (n_\nu A_\rho), \quad v^\mu - h^{\mu\nu} A_\nu,$$

are all invariant under Milne boosts (as $h^{\mu\nu}$ is already invariant). This is a necessary and sufficient condition, provided that we do not endow the quantum field Ψ with transformation properties under the boost. Since the Milne transformations of v^μ and $h^{\mu\nu}$ are fixed, we can only modify the transformation of A_μ . Then the unique redefinition of A_μ which leaves this scalar and vector invariant is

$$(A')_\mu = A_\mu + P_\mu^\nu \psi_\nu - \frac{1}{2} n_\mu h^{\alpha\beta} \psi_\alpha \psi_\beta + n_\mu \frac{g_s}{4m} \varepsilon^{\nu\rho\sigma} \partial_\nu (n_\rho P_\sigma^\alpha \psi_\alpha). \quad (2.32)$$

Putting the pieces together, the theory (2.29) with a magnetic moment is invariant under coordinate reparameterizations, $U(1)$ gauge transformations, and Milne boosts provided that we modify the Milne transformation of A_μ to be (2.32) rather than (2.13c).

Before going on, consider rescaling the gauge field so that Ψ has charge 1. Then the action of the Milne boost is

$$(\bar{A}')_\mu = \bar{A}_\mu + m P_\mu^\nu \psi_\nu - \frac{m}{2} n_\mu \psi^2 + n_\mu \frac{g_s}{4} \varepsilon^{\nu\rho\sigma} \partial_\nu (n_\rho P_\sigma^\alpha \psi_\alpha). \quad (2.33)$$

If one takes the $m \rightarrow 0$ limit (as was used to great effect to study lowest Landau level physics in [6]), one must rescale A_μ this way in order for the theory (2.25) and the transformation laws to be non-singular.

2.7 The relation to Son's non-relativistic covariance

Ever since a paper with Wingate in 2005 [3], Son has progressively developed a notion of non-relativistic “general covariance,” which should be regarded as a definition of invariance under coordinate reparameterization for Galilean-invariant field theories. Unfortunately, as we mentioned in the Introduction, his transformation laws are not defined in a coordinate-independent way. The three major highlights of this development since [3] may be found in [4–6]. We also refer the reader to [16] for some applications of this machinery.

In 2008 [4], Son first wrote down his “general covariance” in terms of the action of infinitesimal reparameterizations of space and time, and showed that this invariance naturally appears in Schrödinger holography. He also showed that the free field theory in (2.24) is covariant in this sense. Five years later, Son observed [5] that his construction is related to Newton-Cartan geometry. In the same paper he introduced the magnetic moment g_s and derived modified transformation laws so that the theory with g_s is invariant under spacetime-dependent reparameterizations of space. Most recently in [6], Son and collaborators have derived the infinitesimal transformations so that the theory with g_s is invariant under reparameterizations of space and time. They also showed how all of these transformations can be understood in a coordinate-independent way, modulo those of A_μ for which they require some choice of coordinates.

For our third and final sanity check, we will show how our proposal for covariance reduces to Son's upon gauge-fixing the Milne symmetry. To do so, we will consider the theory with nonzero g_s . The relation with $g_s = 0$ may be obtained by simply substituting $g_s \rightarrow 0$ in what follows. We first recall the result of [6] for the variations of $(\Phi, g_{ij}, \beta_i, A_0, A_i, \Psi)$ under a coordinate reparameterization ζ^μ and $U(1)$ gauge transformation Λ which leave the action (2.28) invariant. They are

$$\begin{aligned}
\delta\Phi &= \zeta^\mu \partial_\mu \Phi + \beta_i \dot{\zeta}^i - \dot{\zeta}^0, \\
\delta\beta_i &= \zeta^\mu \partial_\mu \beta_i + \beta_j \partial_i \zeta^j - \partial_i \zeta^0 - \beta_i (\dot{\zeta}^0 - \beta_j \dot{\zeta}^j), \\
\delta g_{ij} &= \zeta^\mu \partial_\mu g_{ij} + g_{kj} \partial_i \zeta^k + g_{ik} \partial_j \zeta^k + (\beta_i g_{jk} + \beta_j g_{ik}) \dot{\zeta}^k, \\
\delta A_0 &= \zeta^\mu \partial_\mu A_0 + A_\mu \dot{\zeta}^\mu - \frac{g_s}{4m} \epsilon^{ij} \left[\tilde{\partial}_i (g_{jk} \dot{\zeta}^k) + \dot{\beta}_i g_{jk} \dot{\zeta}^k \right] + \dot{\Lambda}, \\
\delta A_i &= \zeta^\mu \partial_\mu A_i + A_\mu \partial_i \zeta^\mu + e^\Phi g_{ij} \dot{\zeta}^j + \frac{g_s}{4m} \beta_i \epsilon^{jk} \left[\tilde{\partial}_j (g_{kl} \dot{\zeta}^l) + \dot{\beta}_j g_{kl} \dot{\zeta}^l \right] + \partial_i \Lambda, \\
\delta\Psi &= \zeta^\mu \partial_\mu \Psi + im\Lambda\varphi,
\end{aligned} \tag{2.34}$$

where $\tilde{\partial}_i = \partial_i + \beta_i \partial_0$, a dot refers to a derivative with respect to x^0 , and our convention for ζ^μ is minus that of [6]. Note that Ψ is the only field which transforms like a tensor under reparameterizations.

We would like to recover (2.34) from our construction. To do so, we first observe that the theory (2.28) they write down is of the manifestly covariant form (2.29) upon the

identification

$$\begin{aligned}
n_\mu dx^\mu &= e^{-\Phi}(dx^0 - \beta_i dx^i), \\
h^{\mu\nu} \partial_\mu \otimes \partial_\nu &= \beta^2 \partial_0 \otimes \partial_0 + \beta^i (\partial_0 \otimes \partial_i + \partial^i \otimes \partial_0) + g^{ij} \partial_i \otimes \partial_j, \\
v^\mu \partial_\mu &= e^\Phi \partial_0, \\
h_{\mu\nu} dx^\mu \otimes dx^\nu &= g_{ij} dx^i \otimes dx^j,
\end{aligned} \tag{2.35}$$

where $\beta^i = g^{ij} \beta_j$. As we showed in the previous Subsection, the covariant theory (2.29) is invariant under coordinate reparameterizations, $U(1)$ gauge transformations, and modified Milne boosts (2.32). The infinitesimal form of those transformations under a variation $\chi = (\zeta^\mu, \psi_\mu, \Lambda)$ is

$$\begin{aligned}
\delta_\chi n_\mu &= \zeta^\nu \partial_\nu n_\mu + n_\nu \partial_\mu \zeta^\nu, \\
\delta_\chi h^{\mu\nu} &= \zeta^\rho \partial_\rho h^{\mu\nu} - h^{\mu\rho} \partial_\rho \zeta^\nu - h^{\nu\rho} \partial_\rho \zeta^\mu, \\
\delta_\chi v^\mu &= \zeta^\nu \partial_\nu v^\mu - v^\nu \partial_\nu \zeta^\mu + h^{\mu\nu} \psi_\nu, \\
\delta_\chi h_{\mu\nu} &= \zeta^\rho \partial_\rho h_{\mu\nu} + h_{\mu\rho} \partial_\nu \zeta^\rho + h_{\nu\rho} \partial_\mu \zeta^\rho - (n_\mu P_\nu^\rho + n_\nu P_\mu^\rho) \psi_\rho, \\
\delta_\chi A_\mu &= \zeta^\nu \partial_\nu A_\mu + A_\nu \partial_\mu \zeta^\nu + P_\mu^\nu \psi_\nu + \frac{g_s}{4m} n_\mu \varepsilon^{\nu\rho\sigma} \partial_\nu (n_\rho P_\sigma^\alpha \psi_\alpha) + \partial_\mu \Lambda, \\
\delta_\chi \Psi &= \zeta^\mu \partial_\mu \Psi + im \Lambda \Psi.
\end{aligned} \tag{2.36}$$

Now we come to the crux. Given (2.35), we can completely fix the Milne symmetry by demanding that $v^i = 0$. Under an arbitrary reparameterization ζ^μ , we must also perform a Milne boost to keep $v^i = 0$, which fixes the boost parameter ψ_μ in terms of ζ^ν . We have

$$\delta_\chi v^i = -e^\Phi \dot{\zeta}^i + h^{iv} \psi_\nu = 0, \tag{2.37}$$

which then implies

$$h^{iv} \psi_\nu = \beta^i \psi_0 + g^{ij} \psi_j = e^\Phi \dot{\zeta}^i, \tag{2.38}$$

or equivalently $P_i^\mu \psi_\mu = e^\Phi g_{ij} \dot{\zeta}^j$ (note also that $P_0^\mu = 0$).

We will now show that the infinitesimal transformations (2.36) subject to this constraint lead to exactly Son's non-relativistic "general covariance" (2.34). We begin with Φ , using that we can write the variation of v^0 in two ways,

$$\delta_\chi v^0 = e^\Phi \delta_\chi \Phi = \zeta^\mu \partial_\mu v^0 - v^0 \dot{\zeta}^0 + h^{0\nu} \psi_\nu. \tag{2.39}$$

Using that $v^0 = e^\Phi$ and

$$h^{0\nu} \psi_\nu = \beta^2 \psi_0 + \beta^i \psi_i = \beta_i (\beta^i \psi_0 + g^{ij} \psi_j) = e^\Phi \beta_i \dot{\zeta}^i, \tag{2.40}$$

we find

$$\delta_\chi \Phi = \zeta^\mu \partial_\mu \Phi - \dot{\zeta}^0 + \beta_i \dot{\zeta}^i, \tag{2.41}$$

which exactly reproduces the variation of Φ in (2.34). Similarly, we write the variation of

β_i in terms of variations of n_i and Φ to obtain

$$\begin{aligned}\delta_\chi \beta_i &= -\delta_\chi \left(e^\Phi n_i \right) = -e^\Phi \delta_\chi n_i + \beta_i = \delta_\chi \Phi \\ &= -e^\Phi \left(\xi^\mu \partial_\mu (-e^{-\Phi} \beta_i) + e^{-\Phi} \partial_i \xi^0 - e^{-\Phi} \beta_j \partial_i \xi^j \right) + \beta_i \xi^\mu \partial_\mu \Phi - \beta_i \left(\dot{\xi}^0 + \beta_j \dot{\xi}^j \right) \\ &= \xi^\mu \partial_\mu \beta_i - \partial_i \xi^0 + \beta_j \partial_i \xi^j - \beta_i \left(\dot{\xi}^0 - \beta_j \dot{\xi}^j \right),\end{aligned}\quad (2.42)$$

which is the variation of β_i in (2.34). Because $\delta_\chi v^i = 0$ under these constrained transformations, it also follows from $h_{\mu\nu} v^\nu = 0$ that $\delta_\chi h_{0\mu} = 0$. The only part of $h_{\mu\nu}$ which varies is its spatial part, giving

$$\begin{aligned}\delta_\chi g_{ij} &= \delta_\chi h_{ij} = \xi^\mu \partial_\mu g_{ij} + g_{ik} \partial_j \xi^k + g_{jk} \partial_i \xi^k - \left(n_i P_j^\mu + n_j P_i^\mu \right) \psi_\mu \\ &= \xi^\mu \partial_\mu g_{ij} + g_{ik} \partial_j \xi^k + g_{jk} \partial_i \xi^k + \left(\beta_i g_{jk} + \beta_j g_{ik} \right) \dot{\xi}^k,\end{aligned}\quad (2.43)$$

coinciding with the variation in (2.34). We are then left with A_μ . Substituting $n_\mu dx^\mu = e^{-\Phi} (dx^0 - \beta_i dx^i)$ and $P_i^\mu \psi_\mu = e^\Phi g_{ij} \dot{\xi}^j$ into the infinitesimal variation of A_μ in (2.36) immediately gives the variations of A_0 and A_i given in (2.34).

We see that the coordinate reparameterizations of Son's non-relativistic "general covariance" [4] (and its most recent incarnation in [6]) are nothing more than the infinitesimal reparameterizations acting on a Newton-Cartan structure subject to invariance under Milne boosts (2.36) under the constraint that $v^i = 0$.

This is not the whole story. After writing down an action of the form (2.28) and infinitesimal symmetries (2.34), the authors of [6] restore the most general configuration for the velocity v^μ . The most general v^μ consistent with the background for $(n_\mu, h^{\mu\nu})$ appearing in (2.28),

$$n_\mu dx^\mu = e^{-\Phi} (dx^0 - \beta_i dx^i), h^{\mu\nu} \partial_\mu \otimes \partial_\nu = \beta^2 \partial_0 \otimes \partial_0 + \beta^i (\partial_0 \otimes \partial_i + \partial_i \otimes \partial_0) + g^{ij} \partial_i \otimes \partial_j,$$

can be parameterized by the spatial covector u_i to give

$$v^\mu \partial_\mu = e^\Phi \partial_0 + e^\Phi \left(\beta^i u_i \partial_0 + u^i \partial_i \right), \quad (2.44)$$

where $u^i = g^{ij} u_j$, which in turn leads to

$$h_{\mu\nu} dx^\mu \otimes dx^\nu = g_{ij} dx^i \otimes dx^j - e^\Phi u_i \left(n_\mu dx^\mu \otimes dx^i + dx^i \otimes n_\mu dx^\mu \right) + e^{2\Phi} u^2 n_\mu n_\nu dx^\mu \otimes dx^\nu. \quad (2.45)$$

The authors of [6] then claim that the inhomogeneous infinitesimal transformations (2.34) are a consequence of a tensorial variation under coordinate reparameterization, e.g. $\delta h_{\mu\nu} = \mathcal{L}_\xi h_{\mu\nu}$. From this they obtain the infinitesimal variations of u_i and u^2 , which they use to

construct a new, twisted $U(1)$ connection \tilde{A}_μ from A_μ , u_i , and u^2 . It is⁸

$$\begin{aligned}\tilde{A}_0 &= A_0 - \frac{1}{2}e^\Phi u^2 - \frac{g_s}{4m}\epsilon^{ij}(\tilde{\partial}_i u_j + \dot{\beta}_i u_j) , \\ \tilde{A}_i &= A_i + e^\Phi u_i + \frac{1}{2}e^\Phi u^2 \beta_i + \frac{g_s}{4m}\beta_i \epsilon^{jk}(\tilde{\partial}_j u_k + \dot{\beta}_j u_k) .\end{aligned}\tag{2.46}$$

This connection has the virtue that it transforms as a one-form under their infinitesimal variations

$$\delta \tilde{A}_\mu = \mathcal{L}_{\tilde{\zeta}} \tilde{A}_\mu + \partial_\mu \Lambda .\tag{2.47}$$

They then claim that the generating functional is a functional of $(n_\mu, h^{\mu\nu}, v^\mu, \tilde{A}_\mu)$, in such a way that it is invariant under redefinitions of \tilde{A}_μ and v^μ that leave A_μ invariant.

How do we understand these results in light of our construction? There is no covector in the Newton-Cartan data $(n_\mu, h^{\mu\nu}, v^\mu, A_\mu)$ by which we can covariantly redefine A_μ to give something like \tilde{A}_μ . That is, \tilde{A}_μ cannot be constructed from the Newton-Cartan structure without picking a coordinate system.

Nevertheless there is a way that we can make sense of \tilde{A}_μ . The covector u_i parameterizes an arbitrary Milne boost,

$$\psi_\mu dx^\mu = e^\Phi u_i dx^i .\tag{2.48}$$

That is, the Milne variation of $v^\mu \partial_\mu = e^\Phi \partial_0$ under this boost is

$$(v')^\mu \partial_\mu = (v^\mu + h^{\mu\nu} \psi_\nu) \partial_\mu = e^\Phi \partial_0 + e^\Phi (\beta^i u_i \partial_0 + u^i \partial_i) ,\tag{2.49}$$

which coincides with the velocity (2.44), and in the same way the Milne boost of $h_{\mu\nu}$ coincides with the expression in (2.45). The Milne boost of A_μ , (2.32), gives

$$\begin{aligned}(A')_0 &= A_0 + P_0^\mu \psi_\mu - \frac{1}{2}n_0 \psi^2 + n_0 \frac{g_s}{4m} \epsilon^{\mu\nu\rho} \partial_\mu (n_\nu P_\rho^\sigma \psi_\sigma) \\ &= A_0 - \frac{1}{2}e^\Phi u^2 - \frac{g_s}{4m} \epsilon^{ij}(\tilde{\partial}_i u_j + \dot{\beta}_i u_j) = \tilde{A}_0 ,\end{aligned}\tag{2.50}$$

and similarly we find $(A')_i = \tilde{A}_i$. So \tilde{A}_μ is just the Milne-boosted A_μ , and redefinitions of \tilde{A}_μ and v^μ which leave A_μ intact are shifts of the u_i , which we recognize as Milne boosts. In this sense, the authors of [6] agree with our proposal: when they demand invariance under redefinitions of \tilde{A}_μ and v^μ that leave A_μ unchanged, they effectively demand Milne-invariance.

Let us summarize. First, the infinitesimal reparameterizations appearing in Son's non-relativistic "general covariance" are the infinitesimal reparameterizations/Milne boosts in Newton-Cartan geometry subject to the condition $v^i = 0$. Second, the new gauge field \tilde{A}_μ appearing in [5, 6] is the Milne-boosted gauge field where initially $v^i = 0$. Third, the condition introduced in [6] that the generating functional W should be equal for different choices of \tilde{A}_μ and v^μ which leave A_μ intact is essentially our condition that W is invariant under Milne boosts. Finally, the formalism of [6] is almost, but not quite fully covariant.

⁸The expression for \tilde{A}_i in [6] agrees with ours, insofar as they ignored $\mathcal{O}(\beta^2)$ terms.

As an intermediate step in their analysis, they require the variations of A_μ in (2.34) and the boost parameter u_i , both of which are inherently non-covariant.

2.8 An excursus on Wilson lines

A corollary of the results above is the following. Suppose that we gauge the particle number symmetry, as in Galilean-invariant electromagnetism. See Section 3.3 for a discussion of both Galilean-invariant electromagnetism and gravitation. Since the particle number gauge field is not invariant under Milne boosts, any Wilson loop of A_μ will also fail to be invariant under Milne boosts. However, all is not lost. Consider a Wilson loop where the tangent vector t^μ to the loop is always timelike. Here, timelike means that $n_\mu t^\mu$ is everywhere nonzero. This is a somewhat stringent condition, insofar as it restricts us to consider Wilson lines rather than loops. (Although we can form timelike loops around an imaginary-time circle in thermal field theory.)

Nevertheless, let us proceed by normalizing the tangent vector such that $n_\mu t^\mu = 1$. We define $t_\mu = h_{\mu\nu} t^\nu$ and $t^2 = t_\mu t^\mu$. Since t^μ is a tangent vector, it does not transform under Milne boosts, but t_μ and t^2 do. It is easy to show that the combination $t_\mu - \frac{1}{2}n_\mu t^2$ transforms in exactly opposite the way that A_μ transforms, i.e.

$$\Delta_\psi \left(t_\mu - \frac{1}{2}n_\mu t^2 \right) = -P_\mu^\nu \psi_\nu + \frac{1}{2}n_\mu h^{\nu\rho} \psi_\nu \psi_\rho, \quad (2.51)$$

Then we can use this tensor to form a Milne-invariant $U(1)$ connection on the Wilson line. So the Milne-invariant Wilson line along a curve \mathcal{C} is given by

$$\mathcal{W}[\mathcal{C}] = \mathcal{P} \exp \left\{ i \int_{\mathcal{C}} (P[A] + \mathfrak{T}) \right\}, \quad \mathfrak{T} = \left(t_\mu - \frac{1}{2}n_\mu t^2 \right) P[dx^\mu]. \quad (2.52)$$

Unfortunately, we have not been able to find a Milne-invariant version of A_μ along the Wilson line when the magnetic moment g_s is nonzero.

All of this raises the following question. Suppose that we put a Galilean-invariant theory on a lattice and gauge the particle number symmetry. What is the analogue of the Wilson effective action for A_μ ? One strange but logical possibility is that, in order for such a theory to have a good continuum limit, pairwise-separated points on the lattice (connected by a link on which a particle number link variable lives) must be separated in Euclidean time. This would allow for a boost-invariant redefinition of the link variables, while no such redefinition seems to be available on a rectangular lattice.

2.9 Tangent space formulation

We would like to deduce an equivalent formulation of Newton-Cartan structure and Milne boosts in terms of tensors and the spin connection on the tangent bundle $T\mathcal{M}$. Recall how this works for Riemannian geometry. Here one has a positive-definite non-degenerate metric g on spacetime, and derivatives are taken using the Levi-Civita connection constructed from g . $T\mathcal{M}$ is a $GL(d)$ bundle, which in any coordinate patch can be described with the frame fields β_A^μ and their inverse $(\beta^{-1})_\mu^A$, where $A, B = 1, \dots, d$. The statement that $T\mathcal{M}$ is a $GL(d)$ bundle means that the transition maps that relate the

frame fields in two overlapping coordinate patches are valued in $GL(d)$. The metric $g_{\mu\nu}$ induces a metric on the fibers of $T\mathcal{M}$ via $g_{AB} \equiv \beta_A^\mu \beta_B^\nu g_{\mu\nu}$, and the connection $\Gamma^\mu_{\nu\rho} dx^\rho$ induces a spin connection $\omega^A_{B\mu} dx^\mu$ by

$$\mathring{D}_\mu(\beta^{-1})^A_\nu \equiv \partial_\mu(\beta^{-1})^A_\nu - \Gamma^\rho_{\nu\mu}(\beta^{-1})^A_\rho + \omega^A_{B\mu}(\beta^{-1})^B_\nu = 0, \quad (2.53)$$

where \mathring{D}_μ refers to the spin covariant derivative on $T\mathcal{M}$. This gives

$$\omega^A_{B\mu} = (\beta^{-1})^A_\nu D_\mu \beta_B^\nu, \quad (2.54)$$

where here D_μ only acts on the spacetime index of β_B^ν . Equivalently, $\Gamma^\mu_{\nu\rho}$ is determined from the frame fields and the spin connection. Here $\Gamma^\mu_{\nu\rho}$ is the part of the connection which acts on spacetime indices, and the spin connection $\omega^A_{B\mu}$ the part which acts on tangent space indices. One can restrict the frame fields to be orthonormal with respect to the metric g , so that the induced metric on $T\mathcal{M}$ is the Euclidean metric on \mathbb{R}^d ,

$$g_{AB} = \beta_A^\mu \beta_B^\nu g_{\mu\nu} = \delta_{AB}. \quad (2.55)$$

Similarly, one can restrict the frame in the pseudo-Riemannian case so that induced metric is the Minkowski metric. In this case the frame fields are called a *vielbein* and are denoted as E^μ_A and their inverse as e^A_μ . The transition maps which preserve this condition are valued in $O(d) \subset GL(d)$, and so $T\mathcal{M}$ reduces to an $O(d)$ bundle. Then on the tangent space, the metric is an invariant tensor of $O(d)$ which descends to a covariantly constant tensor on \mathcal{M} . By (2.53) and the constancy of g , we have $\mathring{D}_\mu \eta_{AB} = 0$ which implies that $\delta_{C[A} \omega^C_{B]\mu} = 0$, so that the connection one-form ω^A_B is valued in $\mathfrak{o}(d)$. So holonomies of tensor fields are valued in $O(d)$.

Similarly, for a $d = 2n$ -dimensional Kähler manifold \mathcal{M} , one equips \mathcal{M} with a covariantly constant metric g and complex structure J^μ_ν . We can locally choose frame fields in a holomorphic basis, labeling them as E^μ_i or $E^\mu_{\bar{i}}$ with $i, \bar{i} = 1 \dots, n$, so that the metric on tangent space is $\delta_{i\bar{j}}$ and the nonzero components of the induced complex structure are $i\delta^i_j$ and $-i\delta^{\bar{i}}_{\bar{j}}$. The transition maps which preserve this condition are valued in $U(n) \subset O(2n)$, and so in this instance $T\mathcal{M}$ further reduces to a $U(n)$ bundle. The metric and complex structure on tangent space are just invariant tensors of $U(n)$, the spin connection is now valued in $\mathfrak{u}(n)$, and holonomies in $U(n)$.

What is the corresponding situation for our local Galilean invariance? Our approach is to determine the correct formulation by the same logic we reviewed above. We start with Newton-Cartan geometry and Milne/ $U(1)$ -invariance on \mathcal{M} and reduce the structure group on $T\mathcal{M}$ from $GL(d)$ to the smallest possible subgroup. Our results have some overlap and variance with those obtained in [1, 2, 17], as we discuss at the end of the Subsection. We will denote the restricted frame as F^μ_A and the coframe as f^A_μ , to emphasize that since there is no (pseudo-)Riemannian metric, the frame is *not* a vielbein in the sense of (2.55).

Recall that in Newton-Cartan geometry the tensors $(n_\mu, h^{\mu\nu})$ are covariantly constant. We can then further restrict our choice of frame fields to be a *Galilei frame*, which we notate F^μ_A and the coframe as f^A_μ . We restrict $n_\mu = f^0_\mu$ and $h^{\mu\nu} = \delta^{ij} F^\mu_i F^\nu_j$. That is, we restrict to consider frames where the n_A induced by n_μ is $n_A = \delta^0_A$ and h^{AB} is $h^{ij} = \delta^{ij}$.

The transition maps that preserve these conditions are valued in the *Principal Galilean group*, $PGal(d)$. It is a semi-direct product $O(d-1) \ltimes \mathbb{R}^{d-1}$ which is faithfully represented by matrices of the form

$$M = \begin{pmatrix} 1 & 0 \\ K & R \end{pmatrix}, \quad \{K \in \mathbb{R}^{d-1}, R \in O(d-1)\}, \quad (2.56)$$

which acts on the coframe $\begin{pmatrix} n_\mu \\ f_\mu^i \end{pmatrix}$ via right multiplication, and the frame $\begin{pmatrix} F_0^\mu & F_i^\mu \end{pmatrix}$ via inverse left multiplication. So $T\mathcal{M}$ reduces to a $PGal(d)$ bundle, and the spin connection ω^A_B is valued in the algebra of $PGal(d)$, and so has nonzero components ω^i_j and ω^i_0 , with $\delta^{k[i}\omega^{j]}_k = 0$. Under an infinitesimal $PGal(d)$ rotation

$$M_v = \exp \left(-i \begin{pmatrix} 0 & 0 \\ v^i_0 & v^i_j \end{pmatrix} \right), \quad v^{ij} = -v^{ji}, \quad (2.57)$$

the coframe and spin connection vary as

$$\begin{aligned} \delta_v f_\mu^0 &= 0, & \delta_v f_\mu^i &= -v^i_A f_\mu^A, \\ \delta_v \omega^i_{A\mu} &= \partial_\mu v^i_A + \omega^i_{k\mu} v^k_A - v^i_k \omega^k_{A\mu}, \end{aligned} \quad (2.58)$$

What role do Milne boosts and $U(1)$ gauge transformations play in this formulation? Suppose that we use (2.53) to construct the spin connection from the Milne-invariant, but not $U(1)$ -invariant connection Γ_A defined in (2.16). Then by (2.53) it is clear that the spin connection, the frame fields, or both vary under $U(1)$ gauge transformations. Alternatively, if we determine the spin connection from the $U(1)$ -invariant, but not Milne-invariant connection Γ defined in (2.3), then the spin connection, frame fields, or both vary under Milne boosts. In either case, the action of the Milne boosts or $U(1)$ gauge transformations does not come from the action of $PGal(d)$, but instead are additional transformations which act on the tangent space data.

Let us pause and summarize. In terms of the background fields, a Galilean-invariant theory covariantly coupled to \mathcal{M} has a generating functional $W[n_\mu, f_\mu^i, \omega^i_A, v^\mu, A_\mu]$, which is invariant under coordinate reparameterizations, local $PGal(d)$ rotations, $U(1)$ gauge transformations, and Milne boosts. Note that the Milne boosts have nothing to do with action of $PGal(d)$ on the tangent space data. Furthermore, v^μ and A_μ are inert under the action of $PGal(d)$.

Now, there is a way in which the Milne boosts can be understood as arising from the action of $PGal(d)$, at least for $g_s = 0$. Since $v^\mu n_\mu = 1$ and $n_\mu = f_\mu^0$, one may choose the frame to be $\begin{pmatrix} v^\mu & F_i^\mu \end{pmatrix}$, that is one takes the 0-component of the frame to be $F_0^\mu = v^\mu$. This then determines the rest of the coframe in terms of F_i^μ and $h_{\mu\nu}$ via $f_\mu^i = h_{\mu\nu} \delta^{ij} F_i^\nu$ (equivalently, this gives $h_{\mu\nu} = \delta_{ij} f_\mu^i f_\nu^j$). With this choice v^μ is no longer inert under local $PGal(d)$ rotations, (2.56), but transforms as

$$(v')^\mu = v^\mu - F_j^\mu (R^t)^j_i K^i, \quad (2.59)$$

where we have used that for M of the form (2.56), $M^{-1} = \begin{pmatrix} 1 & 0 \\ -R^t K & R^t \end{pmatrix}$. Since $F_j^\mu n_\mu = 0$, v^μ shifts by a transverse vector. Similarly, the spatial coframe shifts as $(f')_\mu^i = K^i n_\mu + R^i_j f_\mu^j$, so that $h_{\mu\nu} = \delta_{ij} f_\mu^i f_\nu^j$ transforms as

$$(h')_{\mu\nu} = h_{\mu\nu} + n_\mu K_i R^i_j f_\nu^j + n_\nu K_i R^i_j f_\mu^j + K^2 n_\mu n_\nu, \quad (2.60)$$

where we notate $K_i = \delta_{ij} K^j$ and $K^2 = K_i K^i$. (2.59) and (2.60) are exactly the transformation laws (2.13a) and (2.13b) for $(v^\mu, h_{\mu\nu})$ under Milne boosts with the identification

$$P_\mu^\nu \psi_\nu = -K_i R^i_j f_\mu^j. \quad (2.61)$$

That is, by choosing a particular Galilei frame in which $F_0^\mu = v^\mu$, the action of the Milne boosts on $(v^\mu, h_{\mu\nu})$ is realized as part of the action of $PGal(d)$ on $T\mathcal{M}$. Since Milne boosts also shift A_μ as (2.13c), $PGal(d)$ must also act on A_μ in this setting. That action can be

efficiently described by combining the coframe with A_μ into a column vector $\begin{pmatrix} n_\mu \\ f_\mu^i \\ A_\mu \end{pmatrix}$, on which $PGal(d)$ acts by right multiplication via matrices of the form

$$M_A = \begin{pmatrix} 1 & 0 & 0 \\ K & R & 0 \\ -\frac{1}{2}K^2 & -K^t R & 1 \end{pmatrix}, \quad \{K \in \mathbb{R}^{d-1}, R \in O(d-1)\}. \quad (2.62)$$

Under an infinitesimal $PGal(d)$ rotation, the coframe and spin connection vary as in (2.58) and A_μ varies as

$$\delta A_\mu = -\delta_{ij} f_\mu^i v_0^j. \quad (2.63)$$

Naïvely this implies that a Newton-Cartan structure necessitates invariance under Milne boosts. However, as we already emphasized previously, this is not the case. The action (2.59) of $PGal(d)$ on v^μ only arose because we restricted the Galilei frame so that $v^\mu = F_0^\mu$. This point is further underscored when we reintroduce the magnetic moment g_s , so that the Milne transformation of A_μ must be modified as in (2.32). Since the modified transformation involves a derivative of the boost parameter ψ_μ , it cannot be realized via any linear action of $PGal(d)$ on A_μ .

We now compare and contrast these results with those appearing in the recent works [1, 17] which also claim to describe the coupling of non-relativistic theories to \mathcal{M} in terms of tangent space data.

1. Strictly speaking, one should not compare our results with those of [1], as those authors consider the coupling of non-relativistic theories without boost-invariance to \mathcal{M} . However, there is some overlap. Suppose that v^μ (and so $h_{\mu\nu}$) is also covariantly constant.⁹ We can then restrict our choice of frame fields to be of the form $\begin{pmatrix} v^\mu & F_i^\mu \end{pmatrix}$ with $h_{\mu\nu} = \delta_{ij} f_\mu^i f_\nu^j$. By assumption, v^μ is covariantly constant, so $T\mathcal{M}$ can be further

⁹Since we are coupling theories without boost-invariance to \mathcal{M} , we no longer require invariance under Milne boosts.

reduced to an $O(d-1)$ bundle, where $O(d-1)$ is embedded in $GL(d)$ via

$$M = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}, \quad \{R \in O(d-1)\}, \quad (2.64)$$

which again acts on the coframe via right multiplication.

This is exactly the spacetime geometry to which [1] couples non-relativistic theories without the Galilean boost symmetry. So in the language of our work, they couple theories to Newton-Cartan geometry $(n_\mu, h^{\mu\nu}, v^\mu)$, for the special case when v^μ is also constant. Since theories without boost invariance do not necessarily possess global symmetries, they do not necessarily include an A_μ , and when they do it is not twisted into the connection $\Gamma^\mu_{\nu\rho}$. In this context, the generating functional W of the theory is a functional $W = W[n_\mu, f_\mu^i, \omega^i_j, v^\mu; A_\mu]$ where A_μ collectively denotes a background gauge field which couples to any global symmetry currents. W is invariant under coordinate reparameterizations, local $O(d-1)$ rotations, and gauge transformations. Equivalently, W is a functional $W = W[n_\mu, h^{\mu\nu}, v^\mu; A_\mu]$ invariant under coordinate reparameterizations and gauge transformations.

2. Unlike [1], the authors of [17] claim to couple Galilean-invariant theories to \mathcal{M} . Their approach is rather different than ours, and we postpone a detailed comparison with our work until Appendix B. For now we give the highlights. While they manifestly realize the rotational and $U(1)$ subgroups of the Galilean symmetry, they impose the Galilean boosts through the addition of a dynamical field u^i . Integrating over u^i enforces the boost symmetry. Already, this should alert the reader that their work presumably makes contact with the physics of spontaneous symmetry breaking, rather than the coupling of a general Galilean-invariant theory to spacetime.

In their construction, quantum fields are coupled to a coframe f_μ^A , a vector-valued one-form Ω_μ^i in which the connection $\omega^i_{0\mu}$ appears, and a connection with components $\omega^i_{j\mu}$ and A_μ . The coframe f_μ^A and the connection coefficients $(\omega^i_{A\mu}, A_\mu)$ transform in the same way as the coframe and connection in our analysis in (2.58). However the Milne boosts do not appear in their setup.

There is another difference between their work and ours. Their auxiliary field u^i appears algebraically in the coframe f_μ^A and A_μ , but through a derivative in Ω_μ^i . Given a local microscopic action in which Ω_μ^i does not appear, u^i is an auxiliary field and may be integrated out to give a new local microscopic action with the same symmetries. However, when the microscopic theory has couplings to Ω_μ^i , u^i appears through derivatives in the action and is not an auxiliary field: integrating over u^i produces a non-local action.

Nevertheless, we find that the construction of [17] can be healed as we describe in Appendix B.4. The resulting geometric structure is equivalent to writing $T\mathcal{M}$ as a $PGal(d)$ bundle and restricting the frame so that the Galilean boosts on $T\mathcal{M}$ become the Milne boosts as in (2.62).

3 Galilean boosts from null reductions

Holographic duality relates quantum gravity on certain spacetimes with boundary to quantum field theories that, roughly speaking, “live” on the boundary. The canonical example of holography is the equivalence between type IIB string theory on $\text{AdS}_5 \times \mathbb{S}^5$ (here AdS_5 is five-dimensional Anti-de-Sitter spacetime) and four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory [18]. There are also holographic dualities that equate quantum gravity on so-called Schrödinger spacetimes with Galilean-invariant field theories [4, 8–10].

Holography has the very useful feature that it dynamically incorporates the coupling of field theories to curved spacetimes. The dual field theory simply couples to the geometry on the boundary of the higher-dimensional spacetime. This geometry is not arbitrary: it must be realized dynamically in a consistent theory of quantum gravity. In this way, holography implicitly answers the question of how to couple Galilean-invariant theories to spacetime.

In this Section we show that our proposal in Subsection 2.3 describes the boundary geometry of asymptotically Schrödinger spacetimes. That is, our proposal is realized holographically. To do so, we first recall that the boundary geometry of Schrödinger spacetimes is a Lorentzian manifold with a null isometry, and second show how the reduction of these manifolds along the isometry leads to a Newton-Cartan structure and Milne invariance.¹⁰

3.1 Manifolds with null isometries, Newton-Cartan structure, and boosts

Consider a $d + 1$ -dimensional Lorentzian manifold \mathcal{M}_{d+1} with metric G and a null isometry generated by $n^M \partial_M$. The geometry of \mathcal{M}_{d+1} is that of a fiber bundle over a d -dimensional base \mathcal{M}_d , where the fibers are either \mathbb{S}^1 or \mathbb{R} depending on whether the integral curves of the null isometry are compact or non-compact. We choose coordinates on \mathcal{M}_{d+1} , $x^M = (x^\mu, x^-)$ so that the null isometry is $n^M \partial_M = \partial_-$, where x^- denotes the affine parameter along integral curves of n , and the components of G are explicitly independent of x^- . The x^μ furnish coordinates on \mathcal{M}_d .

Locally, we can parameterize the most general such G that manifests reparameterization invariance on \mathcal{M}_d along with reparameterizations of x^- of the form $(x')^- = x^- + f(x^\mu)$. It is

$$G = 2n_\mu dx^\mu (dx^- + A_\mu dx^\mu) + h_{\mu\nu} dx^\mu dx^\nu, \quad (3.1)$$

where $h_{\mu\nu}$ is a positive semi-definite tensor of rank $d - 1$. Before going on, let us check that this is indeed the most general such G . In coordinates where $n^M \partial_M = \partial_-$, we must have $G_{--} = 0$, so that G has $\frac{(d+1)(d+2)}{2} - 1 = \frac{d(d+3)}{2}$ independent components. We have parameterized these through the $2d$ independent components of (n_μ, A_μ) and the $\frac{d(d-1)}{2}$ independent components of the degenerate $h_{\mu\nu}$.

¹⁰Our results have some overlap with those of [7, 19]. The former showed how Newton-Cartan structures with $dn = 0$ arises via a null reduction, and the latter considered null reductions of Einstein manifolds.

The inverse of G is

$$G^{-1} = \partial_- \otimes (v^\mu - h^{\mu\nu} A_\nu) \partial_\mu + (v^\mu - h^{\mu\nu} A_\nu) \partial_\mu \otimes \partial_- + h^{\mu\nu} \partial_\mu \otimes \partial_\nu + (A^2 - 2v \cdot A) \partial_- \otimes \partial_-, \quad (3.2)$$

where $A^2 = h^{\mu\nu} A_\mu A_\nu$. Here, $v^\mu \partial_\mu$ is the unique zero-eigenvector of $h_{\mu\nu}$ which (i.) does not have a component along x^- and (ii.) satisfies $n_\mu v^\mu = 1$, and $h^{\mu\nu}$ satisfies

$$h^{\mu\nu} n_\nu = 0, \quad h_{\mu\rho} h^{\nu\rho} = P_\mu^\nu = \delta_\mu^\nu - v^\mu n_\nu. \quad (3.3)$$

The measure is

$$\sqrt{-G} = \sqrt{\det(n_\mu n_\nu + h_{\mu\nu})} = \sqrt{\gamma}. \quad (3.4)$$

The components along \mathcal{M}_d of the Levi-Civita connection Γ_G built from G are

$$(\Gamma_G)^\mu{}_{\nu\rho} = \frac{1}{2} v_A^\mu \partial_{(\nu} n_{\rho)} + \frac{1}{2} h^{\mu\sigma} (\partial_\nu (h_A)_{\rho\sigma} + \partial_\rho (h_A)_{\nu\sigma} - \partial_\sigma (h_A)_{\nu\rho}) = (\tilde{\Gamma}_A)^\mu{}_{\nu\rho}, \quad (3.5)$$

where $v_A^\mu = v^\mu - h^{\mu\nu} A_\nu$ and $(h_A)_{\mu\nu} = h_{\mu\nu} + n_\mu A_\nu + n_\nu A_\mu$. Here we recognize $\tilde{\Gamma}_A$ to be the torsionless part of the Milne-invariant, but not $U(1)$ -invariant connection Γ_A which we defined in (2.16). Because n^M generates an isometry and is a null vector, its covariant derivative under Γ_G , satisfies

$$(D_G)_M n_N = \frac{1}{2} F_{MN}^n, \quad F_{MN}^n = \partial_M n_N - \partial_N n_M, \quad F_{MN}^n n^N = 0, \quad (3.6)$$

where we denote this tensor with an F in analogy with the field strength of a $U(1)$ connection. One can define a torsionful connection Γ_T in such a way that both G and n are covariantly constant, shifting Γ_G as

$$(\Gamma_T)^M{}_{NP} = (\Gamma_G)^M{}_{NP} + \frac{1}{2} G^{MQ} T_{QNP}, \quad (3.7)$$

where T_{QNP} is fully antisymmetric and

$$T_{-\mu\nu} = F_{\nu\mu}^n. \quad (3.8)$$

The components along \mathcal{M}_d of Γ_T are

$$(\Gamma_T)^\mu{}_{\nu\rho} = v_A^\mu \partial_\rho n_\nu + \frac{1}{2} h^{\mu\sigma} (\partial_\nu (h_A)_{\rho\sigma} + \partial_\rho (h_A)_{\nu\sigma} - \partial_\sigma (h_A)_{\nu\rho}) = (\Gamma_A)^\mu{}_{\nu\rho}, \quad (3.9)$$

where Γ_A is the Milne-invariant connection defined in (2.16)

In fact, in (3.1), (3.2), (3.4), and (3.9) we recognize all of the tensor data $(n_\mu, h^{\mu\nu}, v^\mu, A_\mu)$ and the derivative that defines a Newton-Cartan structure. Note that A_μ is the graviphoton of the reduction. This verifies our claim that the Newton-Cartan data automatically arises on the base manifold \mathcal{M}_d . One can also turn our logic around, and build a $d+1$ -dimensional \mathcal{M}_{d+1} from a Newton-Cartan structure on \mathcal{M}_d . This higher-dimensional construction also clears up one nagging aspect of the Newton-Cartan analysis, namely that there was no connection on \mathcal{M} which was simultaneously Milne-invariant and $U(1)$ -

invariant. On \mathcal{M}_{d+1} , $U(1)$ gauge transformations are additive reparameterizations of x^- along \mathcal{M}_d . The gauge variation of the torsionless part of Γ_A , $(\tilde{\Gamma}_A)^\mu{}_{\nu\rho}$, is just the tensorial transformation of $(\Gamma_G)^\mu{}_{\nu\rho}$ under this reparameterization.

The other part of our claim is that Milne boosts naturally arise from the null reduction. To see this, note that the identification of A_μ and $h_{\mu\nu}$ from the metric G (3.1) is not unique. We could just as well have identified

$$(A')_\mu = A_\mu + \Psi_\mu, \quad (h')_{\mu\nu} = h_{\mu\nu} - (n_\mu \Psi_\nu + n_\nu \Psi_\mu), \quad (3.10)$$

for an arbitrary Ψ_μ . That is, $G = 2n_\mu dx^\mu (dx^- + A') + (h')_{\mu\nu} dx^\mu dx^\nu$. Under such a shift, however, we also require that G^{-1} is mapped into a tensor of the same form, which means that

$$h^{\mu\nu}, \quad v^\mu - h^{\mu\nu} A_\nu, \quad A^2 - 2v \cdot A, \quad (3.11)$$

must be invariant under the shift. This fixes Ψ_μ to be of the form

$$\Psi_\mu = P_\mu^\nu \psi_\nu - \frac{1}{2} n_\mu \psi^2, \quad (3.12)$$

and moreover v^μ varies as

$$(v')^\mu = v^\mu + h^{\mu\nu} \psi_\nu. \quad (3.13)$$

Of course this redefinition is just the Milne boost (2.13). So we see that Milne boosts are indeed realized on \mathcal{M}_{d+1} : they correspond to an ambiguity in the identification of the Newton-Cartan data from the higher-dimensional metric G .

From this analysis, it is clear that a different organizing principle is required to obtain a magnetic moment $g_s \neq 0$ from the null reduction and so from holography. We leave this question for future work.

We conclude this Subsection with a study of $T\mathcal{M}_{d+1}$ along the lines of Subsection 2.9. Since G and n are covariantly constant under the derivative defined through Γ_T , we can restrict the frame to be a vielbein E_A^M so that $e_M^0 dx^M = n_\mu dx^\mu$, $E_-^M \partial_M = n^M \partial_M = \partial_-$, and the metric is

$$G = e^0 \otimes e^- + e^- \otimes e^0 + \delta_{ij} e^i \otimes e^j. \quad (3.14)$$

That is, the metric on the tangent space is the flat Minkowski metric where $(0, -)$ are null directions. It almost immediately follows that $T\mathcal{M}_{d+1}$ can be reduced to a $PGal(d)$ bundle over \mathcal{M}_{d+1} , where $PGal(d)$ is embedded into $GL(d+1)$ via matrices of the form

$$M_T = \begin{pmatrix} 1 & 0 & 0 \\ K & R & 0 \\ -\frac{1}{2}K^2 & -K^t R & 1 \end{pmatrix}, \quad \{K \in \mathbb{R}^{d-1}, R \in O(d-1)\}. \quad (3.15)$$

In this way, $PGal(d)$ acts on the coframe $\begin{pmatrix} e_\mu^0 \\ e_\mu^i \\ e_\mu^- \end{pmatrix}$ via right multiplication and the frame $\begin{pmatrix} E_0^\mu & E_i^\mu & E_-^\mu \end{pmatrix}$ via inverse left multiplication.

In Subsection 2.9, we found that the action of the Milne boosts was not a consequence

of the action of $PGal(d)$ on the tangent space data. However, by restricting the Galilei frame further so that $E_0^\mu = v^\mu$, the Milne boosts could be realized through the action of $PGal(d)$, at least for $g_s = 0$. $PGal(d)$ then acted on the coframe and A_μ via (2.62), that is through matrices of the same form as M_T . Can we understand this from our higher-dimensional construction?

There is a natural restriction of the frame on \mathcal{M}_{d+1} which indeed leads to (2.62) upon the null reduction. Restricting the coframe (and so the frame) to be

$$\left(e_M^A dx^M \right) = \begin{pmatrix} n_\mu dx^\mu \\ e_\mu^i dx^\mu \\ dx^- + A \end{pmatrix}, \quad \left(E_A^M \partial_M \right) = \left(v^\mu (\partial_\mu - A_\mu \partial_-) \ F_i^\mu (\partial_\mu - A_\mu \partial_-) \ \partial_- \right), \quad (3.16)$$

where we denote $F_i^\mu = h^{\mu\nu} \delta_{ij} e_\nu^j$, then the action of $PGal(d)$ on the coframe via (3.15)

descends to the action (2.62) on $\begin{pmatrix} n_\mu \\ e_\mu^i \\ A_\mu \end{pmatrix}$. So e_μ^i becomes the spatial coframe f_μ^i in the Newton-Cartan geometry. Similarly, the action (3.15) on the restricted frame (3.16) descends to the action of $PGal(d)$ on the restricted frame $\begin{pmatrix} v^\mu \ F_i^\mu \end{pmatrix}$ in (2.56).

3.2 Using the reduction to construct tensors

Let us briefly return to our discussion of Newton-Cartan geometry in Subsections 2.1 and 2.2. One of the results in Subsection 2.2 was that there was no way to define the covariant derivative that was simultaneously invariant under Milne boosts and $U(1)$ gauge transformations. As a result it is cumbersome to construct Milne/ $U(1)$ -invariant tensorial data out of the background fields. One approach would be to build Milne-invariant tensors from the Milne-invariant derivative defined through (2.16) and the Milne-invariant combinations of background fields $(n_\mu, h^{\mu\nu}, v_A^\mu; (h_A)_{\mu\nu})$, and then afterward deduce $U(1)$ -invariant combinations.

Thankfully, we do not need to determine tensors in that thankless way. We can instead use the embedding of the Newton-Cartan data into a metric G and null isometry n on \mathcal{M}_{d+1} , which automatically incorporates the Milne and $U(1)$ symmetries. It is easy to compute tensors on \mathcal{M}_{d+1} built from G and n , and thereby obtain Milne/ $U(1)$ -invariant tensors from reduction.

This is a particularly simple task when it comes to finding scalars, as we now show. At zeroth order in derivatives, the tensor data on \mathcal{M}_{d+1} is just the metric G , the epsilon tensor $\varepsilon^{M_1 \dots M_{d+1}}$, and the null vector n . There are no scalars, on account of the fact that n is null. At first order in derivatives one can construct tensors from $(D_g)_M n_N$. However, the symmetric part of this tensor vanishes by the fact that n generates an isometry, and so we only have the antisymmetric part dn ,

$$F^n = dn. \quad (3.17)$$

By the isometry condition and n being null, we also have

$$F_{MN}^n n^N = 0,$$

so that in the coordinates (x^μ, x^-) in which $n = \partial_-$ and G is given by (3.1), we have

$$F^n = \frac{1}{2} F_{MN}^n dx^M \wedge dx^N = \frac{1}{2} F_{\mu\nu}^n dx^\mu \wedge dx^\nu. \quad (3.18)$$

At second order in derivatives, one has the Riemann tensor \mathcal{R}_{NPQ}^M , the second derivative of n , $D_{(M} D_{N)} n_P$, and tensors built from two factors of F^n . While there are many tensors that can be formed from this data, there are few scalars. The scalars that can be constructed from the Riemann tensor are the Ricci scalar \mathcal{R} and $\mathcal{R}_{nn} \equiv \mathcal{R}_{MN} n^M n^N$ for \mathcal{R}_{MN} the Ricci tensor. However one can easily show that the isometry implies

$$\mathcal{R}_{nn} = \frac{1}{4} (F^n)^{MN} F_{MN}^n. \quad (3.19)$$

Similarly, all scalars that can be built from the second derivative of n are proportional to $(F^n)^2$. As a result the independent two-derivative scalars are \mathcal{R} and $(F^n)^{MN} F_{MN}^n$.

So far we have considered scalars on \mathcal{M}_{d+1} , which reduce to scalars on \mathcal{M}_d . There are also objects which are not quite scalars, but whose integral over \mathcal{M}_d is invariant under the symmetries of the problem up to boundary terms. Here we follow the discussion of [20], and a similar discussion may be found in [21]. Consider a current \mathcal{X}^M which is identically conserved on \mathcal{M}_{d+1} and which moreover is explicitly independent of x^- . Then its $-$ component transforms under reparameterizations $y = y(x)$ as

$$\mathcal{X}^- \rightarrow \mathcal{X}^M \frac{\partial y^-}{\partial x^M}, \quad (3.20)$$

so that

$$\int d^d x \sqrt{\gamma} \mathcal{X}^- \quad (3.21)$$

is reparameterization-invariant up to a boundary term. In this way, this object is a Chern-Simons term on \mathcal{M}_d .

We can also obtain invariant tensors which include quantum fields on \mathcal{M}_d . For instance, consider a complex field Ψ on \mathcal{M} as in the free-field theory (2.25). We can extend Ψ to a field φ on \mathcal{M}_{d+1} in the coordinates used in (3.1) by letting $\varphi \equiv e^{imx^-} \Psi(x^\mu)$. Note that φ is not invariant under the action of n , but is an eigenfunction thereof,

$$\mathcal{L}_n \varphi = im \varphi. \quad (3.22)$$

Then the free field theory (2.25) is efficiently written in terms of φ as

$$S_{cov} = \int d^d x \sqrt{\gamma} \left\{ \frac{iv^\mu}{2} \Psi^\dagger \overleftrightarrow{D}_\mu \Psi - \frac{h^{\mu\nu}}{2m} D_\mu \Psi^\dagger D_\nu \Psi \right\} = -\frac{1}{2m} \int d^d x \sqrt{-G} G^{MN} \partial_M \varphi^\dagger \partial_N \varphi. \quad (3.23)$$

Along similar lines to free field theory, suppose that we wish to write down the action of a point particle coupled to the Newton-Cartan data on \mathcal{M} . We can deduce the correct Milne/ $U(1)$ -invariant action by starting with a point particle on \mathcal{M}_{d+1} , whose worldline time is parameterized by τ and whose position is given by the fields $X^M(\tau)$. At leading order in gradients, the most general action for a point particle on \mathcal{M}_{d+1} that

couples to G and the null isometry n is

$$S_{pp}^{(d+1)} = \int d\tau \dot{X}^\mu n_\mu f \left(\frac{\sqrt{-G_{MN}(X) \dot{X}^M \dot{X}^N}}{\dot{X}^\rho n_\rho} \right) = \int d\tau \dot{X}^\mu n_\mu f(Y), \quad (3.24)$$

where f is an arbitrary function and we have defined the combination

$$Y = \frac{\sqrt{-(2\dot{X}^- \dot{X}^\mu n_\mu + \dot{X}^\mu \dot{X}^\nu (h_A)_{\mu\nu})}}{\dot{X}^\rho n_\rho}. \quad (3.25)$$

The analogue of (3.22) here is that the momentum along x^- is constant. In this instance that momentum is the constant of motion

$$m = -\frac{\dot{X}^\mu n_\mu}{\sqrt{-(2\dot{X}^- \dot{X}^\mu n_\mu + \dot{X}^\mu \dot{X}^\nu (h_A)_{\mu\nu})}} f'(Y) = -\frac{1}{Y} f'(Y), \quad (3.26)$$

which determines \dot{X}^- as

$$\dot{X}^- = -\frac{\dot{X}^\mu n_\mu (f')^2}{2m^2} - \frac{\dot{X}^\mu \dot{X}^\nu (h_A)_{\mu\nu}}{2\dot{X}^\rho n_\rho}, \quad (3.27)$$

and gives

$$f(Y) = q - \frac{mY^2}{2}, \quad (3.28)$$

for q a constant. The corresponding action on \mathcal{M}_d is not (3.24) evaluated for this profile of \dot{X}^- , but instead its Legendre transform

$$S_{pp} = S_{pp}^{(d+1)} - \int d\tau m \dot{X}^-, \quad (3.29)$$

where one implicitly substitutes (3.27). The equations of motion for the X^μ stemming from the Legendre-transformed action S_{pp} are those that follow from $S_{pp}^{(d+1)}$ when \dot{X}^- is given by (3.27). But this is readily computed to be

$$S_{pp} = \frac{m}{2} \int d\tau \frac{\dot{X}^\mu \dot{X}^\nu h_{\mu\nu}}{\dot{X}^\rho n_\rho} + m \int P[A] + q \int P[n], \quad (3.30)$$

where P refers to the pullback of a form on \mathcal{M} to the worldline, and we recognize the usual $U(1)$ -invariant “electromagnetic” coupling in the $P[A]$ term. Note that the effect of the arbitrary function f is to determine the charge q with which the particle couples to n_μ . Also note that if one chooses τ such that $\dot{X}^\rho n_\rho = 1$, then the first term is effectively the $\frac{1}{2}mv^2$ kinetic energy of a point particle.

3.3 An aside on Galilean-invariant electromagnetism and gravitation

We now have an algorithm to determine $U(1)$ and Milne-invariant tensors via the null reduction. With this technology, there is a toy problem we can efficiently attack: namely,

imagine promoting (a subset of) the Newton-Cartan data to be dynamical fields, and writing down the most general low-derivative effective field theory that describes their dynamics. The two cases we consider are to let A_μ be dynamical, or to let all of the Newton-Cartan data $(n_\mu, h^{\mu\nu}, v^\mu, A_\mu)$ be dynamical. The first case corresponds to Galilean-invariant “electromagnetism” and the second to Galilean-invariant gravitation.

To our knowledge, there are a variety of papers which study this toy problem, none of which derives field equations from the most general two-derivative action consistent with the symmetries of the problem. It seems that the reason for this oversight is that most of the literature on this (admittedly obscure) subject regards a particular set of field equations (known as Newton’s field equations as in [7]) as fundamental, rather than effective field theory. Since we have the technology, we will take a brief detour and clarify this business.

From the previous Subsection, there are no invariant scalars with zero or one derivatives, and there are two independent scalars with two derivatives, \mathcal{R} and $(F^n)^{MN}F_{MN}^n$. (Here we assume that parity is preserved, so that we do not include scalars with an epsilon tensor.) So the most general two-derivative effective action that describes Galilean-invariant electromagnetism or gravitation is

$$S_{Gal} = \int d^d x \sqrt{\gamma} \left\{ \frac{1}{16\pi G} (\mathcal{R} - 2\Lambda) + \frac{1}{4g^2} (F^n)^{MN} F_{MN}^n + \mathcal{O}(\partial^3) \right\}. \quad (3.31)$$

Now we take variations, using (3.1)

$$\begin{aligned} \delta G_{--} &= 0, & \delta G_{-\mu} &= \delta n_\mu, \\ \delta G_{\mu\nu} &= \delta h_{\mu\nu} + n_\mu \delta A_\nu + n_\nu \delta A_\mu + A_\nu \delta n_\mu + A_\mu \delta n_\nu. \end{aligned} \quad (3.32)$$

We implicitly use the constrained variational calculus we describe below in Subsection 5.1, whose main output is that we take the variations of n_μ to be unconstrained, which determines the longitudinal variations of $(h^{\mu\nu}, v^\mu)$. The transverse variations of v^μ and $h^{\mu\nu}$, which we denote as \bar{v}^μ and $\bar{h}^{\mu\nu}$, are unconstrained, and in terms of them we have

$$\delta h_{\mu\nu} = - (n_\mu h_{\nu\rho} + n_\nu h_{\mu\rho}) \delta \bar{v}^\rho - h_{\mu\rho} h_{\nu\sigma} \delta \bar{h}^{\rho\sigma}.$$

We also constrain our variations so that they do not depend on x^- , i.e. the action of \mathcal{L}_n on the variations vanishes.

The variation of \mathcal{R} gives

$$\delta \mathcal{R} = -\mathcal{R}^{MN} \delta G_{MN} + (D_G)^M (D_G)^N \delta G_{MN} - \square_G (G^{MN} \delta G_{MN}), \quad (3.33)$$

so that the second and third terms integrate to boundary terms and do not contribute to

the equations of motion. We then have

$$\begin{aligned} \delta S_{Gal} = - \int d^d x \sqrt{\gamma} \left\{ \frac{\delta G_{MN}}{16\pi G} \left[\mathcal{R}^{MN} + \left(-\frac{\mathcal{R}}{2} + \Lambda \right) G^{MN} \right. \right. \\ \left. \left. + \frac{8\pi G}{g^2} \left((F^n)^{MP} (F^n)^N{}_P - \frac{G^{MN}}{4} (F^n)^2 \right) \right] \right. \\ \left. - \frac{1}{g^2} \delta n_M (D_G)_N (F^n)^{MN} \right\} + (\text{boundary term}). \end{aligned} \quad (3.34)$$

Using $G^{\mu\nu} n_\nu = 0$ and $n^M (F^n)_M{}^N = n_\mu (F^n)^{\mu N} = 0$, the equations of motion for the gauge field are

$$-\frac{1}{\sqrt{\gamma}} \frac{\delta S_{Gal}}{\delta A_\mu} = \frac{1}{8\pi G} \mathcal{R}^\mu{}_- = 0. \quad (3.35)$$

This is the equation of motion for source-free Galilean-invariant electromagnetism.

Now consider Galilean-invariant gravitation. When (3.35) is satisfied, the variation $\delta S_{Gal}/\delta \bar{v}^\mu$ also vanishes by the underlying Milne invariance of the action, so it remains to compute the variations with respect to n_μ and $\bar{h}^{\mu\nu}$. We find

$$\begin{aligned} -\frac{1}{\sqrt{\gamma}} \frac{\delta S_{Gal}}{\delta n_\mu} &= \frac{1}{8\pi G} \left\{ \mathcal{R}^\mu{}_- (A^2 - 2v \cdot A) + \mathcal{R}^\mu{}_\nu v^\nu + \left(-\frac{\mathcal{R}}{2} + \Lambda \right) v^\mu \right\} \\ &\quad + \frac{1}{g^2} \left\{ (F^n)^{\mu M} F_{\nu M}^\nu v^\nu - \frac{v^\mu}{4} (F^n)^2 - (D_G)_M (F^n)^{\mu M} \right\} = 0 \\ \frac{2}{\sqrt{\gamma}} \frac{\delta S_{Gal}}{\delta \bar{h}^{\mu\nu}} &= \frac{1}{8\pi G} \left\{ P_\mu^\rho P_\nu^\sigma \mathcal{R}_{\rho\sigma} + \left(-\frac{\mathcal{R}}{2} + \Lambda \right) h_{\mu\nu} \right\} \\ &\quad + \frac{1}{g^2} \left\{ P_\mu^\rho P_\nu^\sigma F_{\rho M}^\nu (F^n)_{\sigma}{}^M - \frac{1}{4} h_{\mu\nu} (F^n)^2 \right\}. \end{aligned} \quad (3.36)$$

4 “Weyl invariance” and Schrödinger symmetry

In certain field theories the Galilean symmetry is enhanced to its conformal extension, known as the Schrödinger group. Recall that the Schrödinger group has a dilatation subgroup under which time scales twice as much as space, that is Schrödinger-invariant theories are characterized by a dynamical critical exponent $z = 2$. There are other examples of scale-invariant, Galilean-invariant theories with $z \neq 2$. In any case, we would like to understand non-relativistic conformal symmetry in the same way as relativistic conformal symmetry, which we remind the reader is invariance under coordinate reparametrizations as well as under Weyl transformations of the background metric.

4.1 Weyl rescalings of Newton-Cartan geometry

Our proposal is that in order to couple Schrödinger-invariant field theories to curved spacetime, we must couple to a Newton-Cartan structure $(n_\mu, h^{\mu\nu}, v^\mu, A_\mu)$ in a way that is invariant under reparametrizations, Milne boosts, $U(1)$ gauge transformations, and “Weyl” transformations

$$n_\mu \rightarrow e^{2\Omega} n_\mu, \quad h^{\mu\nu} \rightarrow e^{-2\Omega} h^{\mu\nu}, \quad v^\mu \rightarrow e^{-2\Omega} v^\mu, \quad A_\mu \rightarrow A_\mu, \quad (4.1)$$

where Ω is a general function on \mathcal{M} . These rescalings preserve the defining relations $h^{\mu\nu}n_\nu = 0$, &c of the Newton-Cartan structure. There is an immediate generalization of this proposal for Galilean-invariant, scale-invariant theories with $z \neq 2$. Namely, couple to a Newton-Cartan structure so that the theory is invariant under the modified ‘‘Weyl’’ transformations

$$n_\mu \rightarrow e^{z\Omega} n_\mu, \quad h^{\mu\nu} \rightarrow e^{-2\Omega} h^{\mu\nu}, \quad v^\mu \rightarrow e^{-z\Omega} v^\mu, \quad A_\mu \rightarrow A_\mu. \quad (4.2)$$

4.2 Relation to the Schrödinger algebra

In the same spirit as in Subsection 2.4, we would like to perform a couple of sanity checks on this proposal. First, we will recompute the global symmetries of the flat Newton-Cartan structure where we now include the action of Weyl transformations. For $z = 2$, the symmetry algebra should be the Schrödinger algebra.

Collectively denoting an infinitesimal reparameterization, Milne boost, $U(1)$ gauge transformation, and Weyl transformation as $\chi = (\xi^\mu \partial_\mu, \psi_\mu dx^\mu, \Lambda, \Omega)$, the action of δ_χ on the Newton-Cartan background is

$$\begin{aligned} \delta_\chi n_\mu &= \mathcal{L}_\xi n_\mu + z\Omega n_\mu = \xi^\nu \partial_\nu n_\mu + n_\nu \partial_\mu \xi^\nu + z\Omega n_\mu, \\ \delta_\chi h^{\mu\nu} &= \mathcal{L}_\xi h^{\mu\nu} - 2\Omega h^{\mu\nu} = \xi^\rho \partial_\rho h^{\mu\nu} - h^{\mu\rho} \partial_\rho \xi^\nu - h^{\nu\rho} \partial_\rho \xi^\mu - 2\Omega h^{\mu\nu}, \\ \delta_\chi v^\mu &= \mathcal{L}_\xi v^\mu + h^{\mu\nu} \psi_\nu - z\Omega v^\mu = \xi^\nu \partial_\nu v^\mu - v^\nu \partial_\nu \xi^\mu + h^{\mu\nu} \psi_\nu - z\Omega v^\mu, \\ \delta_\chi A_\mu &= \mathcal{L}_\xi A_\mu + P_\mu^\nu \psi_\nu + \partial_\mu \Lambda = \xi^\nu \partial_\nu A_\mu + A_\nu \partial_\mu \xi^\nu + P_\mu^\nu \psi_\nu + \partial_\mu \Lambda. \end{aligned} \quad (4.3)$$

These transformations generate an algebra $[\delta_{\chi_1}, \delta_{\chi_2}] = \delta_{\chi_{[12]}}$ with $\chi_{[12]} = (\xi_{[12]}^\mu \partial_\mu, \psi_{[12]}^\mu dx^\mu, \Lambda_{[12]}, \Omega_{[12]})$ given by

$$\begin{aligned} \xi_{[12]}^\mu &= \mathcal{L}_{\xi_1} \xi_2^\mu - \mathcal{L}_{\xi_2} \xi_1^\mu = \xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu, \\ \psi_{[12]}^\mu &= \mathcal{L}_{\xi_1} \psi_2^\mu - \mathcal{L}_{\xi_2} \psi_1^\mu = \xi_1^\nu \partial_\nu \psi_2^\mu + \psi_2^\nu \partial_\nu \xi_1^\mu - \xi_2^\nu \partial_\nu \psi_1^\mu - \psi_1^\nu \partial_\nu \xi_2^\mu, \\ \Lambda_{[12]} &= \mathcal{L}_{\xi_1} \Lambda_2 - \mathcal{L}_{\xi_2} \Lambda_1 = \xi_1^\mu \partial_\mu \Lambda_2 - \xi_2^\mu \partial_\mu \Lambda_1, \\ \Omega_{[12]} &= \mathcal{L}_{\xi_1} \Omega_2 - \mathcal{L}_{\xi_2} \Omega_1 = \xi_1^\mu \partial_\mu \Omega_2 - \xi_2^\mu \partial_\mu \Omega_1. \end{aligned} \quad (4.4)$$

The global symmetries of the flat Newton-Cartan structure on \mathbb{R}^d are generated by those infinitesimal transformations K for which δ_K annihilates the structure. Recall that the flat structure is specified by $n_\mu dx^\mu = dx^0$, $h^{\mu\nu} \partial_\mu \otimes \partial_\nu = \delta^{ij} \partial_i \otimes \partial_j$, $v^\mu \partial_\mu = \partial_0$, and $A = 0$. It is easy to show that the space of such K is finite-dimensional for $d > 1$ and is spanned by

$$H = (-\partial_0, 0, 0), \quad P_i = (-\partial_i, 0, 0, 0), \quad (4.5a)$$

$$R_{ij} = (x^j \partial_i - x^i \partial_j, 0, 0, 0), \quad K_i = (-x^0 \partial_i, -dx^i, x^i, 0), \quad (4.5b)$$

$$M = (0, 0, 1, 0), \quad D = (zx^0 \partial_0 + x^i \partial_i, 0, 0, -1), \quad (4.5c)$$

$$C = \left(-\frac{z}{2} (x^0)^2 \partial_0 - x^0 x^i \partial_i, -x^i dx^i, \frac{x^2}{2}, x^0 \right). \quad (4.5d)$$

We compute the brackets of these generators by (4.4), and find that they satisfy the

Galilean algebra (2.23) along with the extra commutators of D and C . The latter are given by

$$\begin{aligned} [H, D] &= zH, & [P_i, D] &= P_i, & [K_i, D] &= -(z-1)K_i, \\ [D, C] &= zC, & [H, C] &= D, & [P_i, C] &= -K_i, \end{aligned} \quad (4.6)$$

with all other commutators vanishing. However, from this and (2.23) we have

$$[C, [H, K_i]] + [H, [K_i, C]] + [K_i, [C, H]] = (z-2)K_i, \quad (4.7)$$

so that the Jacobi identity is only satisfied for $z = 2$. For $z = 2$, (2.23) and (4.6) are just the brackets of the Schrödinger algebra expressed in a basis of anti-Hermitian generators. For $z \neq 2$, the symmetry algebra does not contain C : it is the Galilean algebra (2.23) plus the dilatation operator D . This is expected: C corresponds to the Galilean analogue of a special conformal symmetry, which only exists for $z = 2$.

4.3 Conformally coupled free fields

As a second sanity check, we would like to exhibit a free field theory coupled to \mathcal{M} which is invariant under reparameterizations, Milne boosts, $U(1)$ gauge transformations, and now Weyl transformations. So we return to the free field theory of a complex scalar coupled to \mathcal{M} (2.25),

$$S_{cov} = \int d^d x \sqrt{\gamma} \left\{ \frac{iv^\mu}{2} \Psi^\dagger \overleftrightarrow{D}_\mu \Psi - \frac{h^{\mu\nu}}{2m} D_i \Psi^\dagger D_j \Psi \right\}.$$

In Subsection 2.5 we showed that this theory is invariant under reparameterizations, Milne boosts, and $U(1)$ gauge transformations. In order to also be invariant under Weyl transformations, we require $z = 2$ so that v^μ transforms with the same weight as $h^{\mu\nu}$. Note that $\sqrt{\gamma}$ transforms under Weyl transformations as

$$\sqrt{\gamma} \rightarrow e^{(d-1+z)\Omega} \sqrt{\gamma}, \quad (4.8)$$

so S_{cov} is invariant under position-independent Weyl rescalings (4.2) provided that Ψ also transforms as

$$\Psi \rightarrow e^{-\frac{d-1}{2}\Omega} \Psi, \quad (4.9)$$

and the same for Ψ^\dagger . However S_{cov} is obviously not invariant under general Weyl rescalings. The remedy is to add a term to the action which couples Ψ to the background curvature, analogous to the conformal mass coupling in relativistic quantum field theory.

In this instance, this is more than analogy. Recall that we can obtain S_{cov} from a null reduction of the free field action (3.23) in one higher dimension, where φ carries momentum m along the extra null direction. Now, note that in terms of the $d+1$ -dimensional metric in (3.1), the Weyl transformation (4.2) for $z = 2$ is just a higher-dimensional Weyl transformation

$$G \rightarrow e^{2\Omega} (2n_\mu dx^\mu (dx^- + A_\nu dx^\nu) + h_{\mu\nu} dx^\mu dx^\nu). \quad (4.10)$$

As a result, the action of a conformally coupled free field φ carrying momentum m in the null direction reduces to the action of a free Ψ conformally coupled to \mathcal{M} . This is

$$S_{\text{conformal}} = -\frac{1}{2m} \int d^d x \sqrt{g} \left\{ G^{MN} \partial_M \varphi^\dagger \partial_N \varphi + \xi \mathcal{R} \varphi^\dagger \varphi \right\}, \quad \xi = \frac{d-1}{4d}, \quad (4.11)$$

where \mathcal{R} is the Ricci scalar curvature of the higher-dimensional metric G in (3.1). Note also that the scale dimension of a free relativistic scalar in $d+1$ dimensions is $\frac{d-1}{2}$, which is exactly the weight with which Ψ scales here.

5 Currents and Ward identities

As a basic application of our machinery, we now define various symmetry currents conjugate to the Newton-Cartan data $(n_\mu, h^{\mu\nu}, v^\mu, A_\mu)$ and compute Ward identities for them. We express all of the Ward identities in terms of the $U(1)$ -invariant, but not Milne-invariant derivative defined from the connection Γ in (2.3). Our results in Subsections 5.1 and 5.2 have a great deal of overlap with those obtained in [6]. However, there are some differences between the two analyses, as we detail in Subsection 5.2.

5.1 Constrained variations

When defining the various currents and stress tensor, we will vary the generating functional W with respect to the background fields $(n_\mu, h^{\mu\nu}, v^\mu, A_\mu)$. However, these variations cannot be arbitrary: they must be consistent with the relations

$$n_\mu h^{\mu\nu} = 0, \quad n_\mu v^\mu = 1, \quad v^\mu h_{\mu\nu} = 0, \quad h_{\mu\rho} h^{\nu\rho} = P_\mu^\nu.$$

As a result, choosing to let the variations of n_μ be arbitrary, the variations of $(h^{\mu\nu}, v^\mu, h_{\mu\nu})$ are constrained. For instance, we have

$$\delta(n_\mu v^\mu) = v^\mu \delta n_\mu + n_\mu \delta v^\mu = 0, \quad (5.1)$$

from which it follows that

$$\delta v^\mu = -v^\mu v^\nu \delta n_\nu + P_\nu^\mu \delta \bar{v}^\nu, \quad (5.2)$$

where $\delta \bar{v}^\mu$ is unconstrained. Similarly we have

$$\begin{aligned} \delta h^{\mu\nu} &= -(v^\mu h^{\nu\rho} + v^\nu h^{\mu\rho}) \delta n_\rho + P_\rho^\mu P_\sigma^\nu \delta \bar{h}^{\rho\sigma}, \\ \delta h_{\mu\nu} &= -(n_\mu h_{\nu\rho} + n_\nu h_{\mu\rho}) \delta \bar{v}^\rho - h_{\mu\alpha} h_{\nu\beta} \delta \bar{h}^{\alpha\beta}, \end{aligned} \quad (5.3)$$

where $\delta \bar{h}^{\mu\nu}$ is unconstrained.

We define connected correlations of operators through the variations of the generating functional W with respect to the conjugate background fields. We take the gauge field A_μ to be conjugate to the particle number current J^μ . The velocity (or more precisely, the unconstrained variation thereof) is conjugate to momentum \mathcal{P}_μ . The clock covector n_μ is conjugate to the energy current, and the spatial cometric $h^{\mu\nu}$ (again, the unconstrained

variation) to be conjugate to the spatial stress tensor $T_{\mu\nu}$. In an equation, we have

$$\delta W = \int d^d x \sqrt{\gamma} \left\{ \delta A_\mu \langle J^\mu \rangle - \delta \bar{v}^\mu \langle \mathcal{P}_\mu \rangle - \delta n_\mu \langle \mathcal{E}^\mu \rangle - \frac{\delta \bar{h}^{\mu\nu}}{2} \langle T_{\mu\nu} \rangle \right\}, \quad (5.4)$$

where we should remember that $\langle \mathcal{P}_\mu \rangle$ and $\langle T_{\mu\nu} \rangle$ are transverse. That is, the variations $\delta \bar{v}^\mu$ and $\delta \bar{h}^{\mu\nu}$ always come from variations of v^μ , $h^{\mu\nu}$, and $h_{\mu\nu}$ and so appear through the projections $P_\nu^\mu \delta \bar{v}^\nu$ and $P_\rho^\mu P_\sigma^\nu \delta \bar{h}^{\rho\sigma}$. In writing (5.4), we have implicitly incorporated those projectors into the definitions of $\langle \mathcal{P}_\mu \rangle$ and $\langle T_{\mu\nu} \rangle$.

Note that we have chosen to define momentum and spatial stress in such a way that the variations of n_μ are unconstrained. We could have instead constrained the variations of n_μ , which would have resulted in a different definition for the conjugate operators. However all such redefinitions are completely equivalent to (5.4).

5.2 Ward identities for one-point functions

In order to obtain the Ward identities, we require (5.4) as well as the variations of the $(n_\mu, \bar{h}^{\mu\nu}, \bar{v}^\mu, A_\mu)$ under the local symmetries. For now, we will take the magnetic moment g_s of Subsection 2.6 to vanish, and restore it below in Subsection 5.6. Under an infinitesimal reparameterization, Milne boost, and $U(1)$ gauge transformation $\chi = (\xi^\mu, \psi_\mu, \Lambda)$, (2.19) gives

$$\delta_\chi n_\mu = \mathcal{L}_\xi n_\mu = -F_{\mu\nu}^n \xi^\nu + D_\mu (\xi^\nu n_\nu), \quad (5.5a)$$

$$\begin{aligned} \delta_\chi \bar{h}^{\mu\nu} &= P_\rho^\mu P_\sigma^\nu \delta_\chi h^{\rho\sigma} = P_\rho^\mu P_\sigma^\nu \mathcal{L}_\xi h^{\rho\sigma} = P_\rho^\mu P_\sigma^\nu (\xi^\alpha \partial_\alpha h^{\rho\sigma} - h^{\alpha\sigma} \partial_\alpha \xi^\rho - h^{\rho\alpha} \partial_\alpha \xi^\sigma) \\ &= P_\rho^\mu P_\sigma^\nu (\xi^\alpha D_\alpha h^{\rho\sigma} - h^{\alpha\sigma} D_\alpha \xi^\rho - h^{\rho\alpha} D_\alpha \xi^\sigma) + P_\rho^\mu P_\sigma^\nu (T^\rho_{\alpha\beta} h^{\beta\sigma} + T^\sigma_{\alpha\beta} h^{\beta\rho}) \xi^\alpha \\ &= P_\rho^\mu P_\sigma^\nu (\xi^\alpha D_\alpha h^{\rho\sigma} - h^{\alpha\sigma} D_\alpha \xi^\rho - h^{\rho\alpha} D_\alpha \xi^\sigma), \end{aligned} \quad (5.5b)$$

$$\begin{aligned} \delta_\chi \bar{v}^\mu &= P_\nu^\mu \delta_\chi v^\nu = P_\nu^\mu (\mathcal{L}_\xi v^\nu + h^{\nu\rho} \psi_\rho) = P_\nu^\mu (\xi^\rho \partial_\rho v^\nu - v^\rho \partial_\rho \xi^\nu) + h^{\mu\nu} \psi_\nu, \\ &= P_\nu^\mu (\xi^\rho D_\rho v^\nu - v^\rho D_\rho \xi^\nu + T^\nu_{\alpha\beta} \xi^\alpha v^\beta) + h^{\mu\nu} \psi_\nu \\ &= P_\nu^\mu (\xi^\rho D_\rho v^\nu - v^\rho D_\rho \xi^\nu) + h^{\mu\nu} \psi_\nu, \end{aligned} \quad (5.5c)$$

$$\delta_\chi A_\mu = \mathcal{L}_\xi A_\mu + P_\mu^\nu \psi_\nu + D_\mu \Lambda = -F_{\mu\nu}^n \xi^\nu + D_\mu (\xi^\nu A_\nu + \Lambda) + P_\mu^\nu \psi_\nu, \quad (5.5d)$$

where F^n is given by

$$F_{\mu\nu}^n = \partial_\mu n_\nu - \partial_\nu n_\mu,$$

and used that spatial torsion vanishes $P_\nu^\mu T^\nu_{\alpha\beta} = 0$.

We now plug the infinitesimal symmetry variations (5.5) into the variation of W (5.4) to obtain the Ward identities, using that W is invariant under the action of the symmetries

$$\delta_\chi W = 0. \quad (5.6)$$

Schematically, we will use (5.5) and (5.4) to write the variation $\delta_\chi W$ as

$$\delta_\chi W = \int d^d x \sqrt{\gamma} \{ \Lambda \mathcal{J} + h^{\mu\nu} \psi_\mu \mathcal{M}_\nu + \xi^\mu \mathcal{T}_\mu \}, \quad (5.7)$$

from which the Ward identities are simply $\mathcal{J} = 0$, $P_\mu^\nu \mathcal{M}_\nu = 0$, and $\mathcal{T}_\mu = 0$.

The gauge parameters Λ and ξ^μ appear through their derivatives in the symmetry variations (5.5). So to proceed we must integrate variations of the gauge parameters Λ and ξ^μ by parts. We would like to do this efficiently and covariantly, so we first understand how the covariant derivatives integrate by parts. To do this we first compute

$$\begin{aligned}\partial_\mu \sqrt{\gamma} &= \frac{1}{2} \sqrt{\gamma} \gamma^{\nu\rho} \partial_\mu \gamma_{\nu\rho} = \sqrt{\gamma} (v^\nu v^\rho + h^{\nu\rho}) (n_\nu \partial_\mu n_\rho + \partial_\mu h_{\nu\rho}) \\ &= \sqrt{\gamma} \left(v^\nu \partial_\mu n_\nu + \frac{1}{2} h^{\nu\rho} \partial_\mu h_{\nu\rho} \right),\end{aligned}\tag{5.8}$$

where we used $v^\nu v^\rho \partial_\mu h_{\nu\rho} = -v^\nu h_{\nu\rho} \partial_\mu v^\rho = 0$. However, we now note that the expression in brackets is related to the connection $\Gamma^\mu_{\nu\rho}$ and its torsion appearing in (2.3). In particular,

$$\partial_\mu \sqrt{\gamma} = \sqrt{\gamma} \Gamma^\nu_{\nu\mu}.\tag{5.9}$$

Following [6], we define

$$\mathcal{G}_\mu \equiv T^\nu_{\mu\nu} = -F^\nu_{\mu\nu} v^\nu,\tag{5.10}$$

so that for any vector field \mathfrak{v}^μ we have

$$(D_\mu - \mathcal{G}_\mu) \mathfrak{v}^\mu = [\partial_\mu + \Gamma^\nu_{\mu\nu} - (\Gamma^\nu_{\mu\nu} - \Gamma^\nu_{\nu\mu})] \mathfrak{v}^\mu = \frac{1}{\sqrt{\gamma}} \partial_\mu (\sqrt{\gamma} \mathfrak{v}^\mu),\tag{5.11}$$

which gives the integration by parts formula

$$\int d^d x \sqrt{\gamma} (D_\mu - \mathcal{G}_\mu) \mathfrak{v}^\mu = (\text{boundary term}).\tag{5.12}$$

We may now proceed efficiently.

We begin with the $U(1)$ Ward identity, which comes from demanding that the gauge variation of W vanishes,

$$\begin{aligned}\delta_\Lambda W &= \int d^d x \sqrt{\gamma} D_\mu \Lambda \langle J^\mu \rangle = \int d^d x \sqrt{\gamma} \{ (D_\mu - \mathcal{G}_\mu) (\Lambda \langle J^\mu \rangle) - \Lambda (D_\mu - \mathcal{G}_\mu) \langle J^\mu \rangle \} \\ &= - \int d^d x \sqrt{\gamma} \Lambda (D_\mu - \mathcal{G}_\mu) \langle J^\mu \rangle + (\text{boundary term}) = 0.\end{aligned}\tag{5.13}$$

Similarly, the Milne Ward identity follows from demanding $\delta_\psi W = 0$, which immediately gives

$$\delta_\psi W = \int d^d x \sqrt{\gamma} h^{\mu\nu} \psi_\mu (h_{\nu\rho} \langle J^\rho \rangle - \langle \mathcal{P}_\nu \rangle) = 0.\tag{5.14}$$

The variation of W under an infinitesimal reparameterization is a bit more complicated, giving

$$\begin{aligned}\delta_\xi W &= \int d^d x \sqrt{\gamma} \left\{ \xi^\mu \left[(F_{\mu\nu} - A_\mu (D_\nu - \mathcal{G}_\nu)) \langle J^\nu \rangle - (F^\nu_{\mu\nu} - n_\mu (D_\nu - \mathcal{G}_\nu)) \langle \mathcal{E}^\nu \rangle \right. \right. \\ &\quad \left. \left. - D_\nu (v^\nu \langle \mathcal{P}_\mu \rangle) - D_\mu v^\nu \langle \mathcal{P}_\nu \rangle - (D_\nu - \mathcal{G}_\nu) (\langle T_{\mu\rho} \rangle h^{\nu\rho}) \right] \right. \\ &\quad \left. + (D_\mu - \mathcal{G}_\mu) [\xi^\nu (A_\nu \langle J^\mu \rangle - n_\nu \langle \mathcal{E}^\mu \rangle + h^{\mu\rho} \langle T_{\nu\rho} \rangle) \right. \\ &\quad \left. + (D_\nu - \mathcal{G}_\nu) (v^\nu \xi^\mu \langle \mathcal{P}_\mu \rangle) \right\} = 0.\end{aligned}\tag{5.15}$$

The last two lines give boundary terms by our integration by parts formula (5.12), and so the reparameterization Ward identity comes from the first two lines. To better understand it, we use the $U(1)$ Ward identity along with that $\langle \mathcal{P}_\mu \rangle$ is transverse to v^μ , so that (5.15) becomes

$$(D_\nu - \mathcal{G}_\nu) \langle T_\mu{}^\nu \rangle - n_\mu (D_\nu - \mathcal{G}_\nu) \langle \mathcal{E}^\nu \rangle + D_\nu (v^\nu \langle \mathcal{P}_\mu \rangle) - v^\nu D_\mu \langle \mathcal{P}_\nu \rangle = F_{\mu\nu} \langle J^\nu \rangle - F_{\mu\nu}^n \langle \mathcal{E}^\nu \rangle, \quad (5.16)$$

where we raise indices with $h^{\mu\nu}$. It is instructive to decompose this into longitudinal and transverse components. Combining those with (5.13) and (5.14) we obtain the full set of $U(1)$, Milne, and reparameterization Ward identities for $(J^\mu, \mathcal{P}_\mu, \mathcal{E}^\mu, T_{\mu\nu})$. They are

$$\begin{aligned} (D_\mu - \mathcal{G}_\mu) \langle J^\mu \rangle &= 0, \\ \langle \mathcal{P}_\mu \rangle &= h_{\mu\nu} \langle J^\nu \rangle, \\ (D_\mu - \mathcal{G}_\mu) \langle \mathcal{E}^\mu \rangle &= v^\mu \left(F_{\mu\nu}^n \langle \mathcal{E}^\nu \rangle - F_{\mu\nu} \langle J^\nu \rangle \right) - \frac{1}{2} (D^\mu v^\nu + D^\nu v^\mu) \langle T_{\mu\nu} \rangle, \\ (D_\nu - \mathcal{G}_\nu) \langle T^{\mu\nu} \rangle &= v^\nu D^\mu \langle \mathcal{P}_\nu \rangle - D_\nu (v^\nu \langle \mathcal{P}^\mu \rangle) + F^\mu{}_\nu \langle J^\nu \rangle - (F^n)^\mu{}_\nu \langle \mathcal{E}^\nu \rangle. \end{aligned} \quad (5.17)$$

There are two minor differences between the final result (5.17) obtained here and that in [6], both of which stem from the same fact. As we explained at the end of Subsection 2.7, in Son's "general covariance" one can combine A_μ , u_i , and u^2 (where we remind the reader that u_i and u^2 are secretly components of $h_{\mu\nu}$ in a particular coordinate system) to obtain a new $U(1)$ connection \tilde{A}_μ (2.46). This new connection has the virtue that it transforms like a one-form under Son's non-relativistic diffeomorphisms.

However, as we pointed out in Subsection 2.7, there is no generally covariant version of \tilde{A}_μ . That is, \tilde{A}_μ does not exist in Newton-Cartan geometry. Our reparameterization Ward identities then differ from those in [6] in that (i.) our field strength is the curvature of A_μ whilst theirs is the curvature of \tilde{A}_μ , and (ii.) our current J^μ is conjugate to A_μ , whilst theirs is conjugate to \tilde{A}_μ .

5.3 Milne variations of currents

The various currents and stress tensor defined in (5.4) have non-trivial transformation laws under Milne boosts. For instance, the momentum current has a Milne variation which is determined by the variations of $h_{\mu\nu}$ and $\langle J^\mu \rangle$ via the Milne Ward identity (5.17). We will presently determine the variations of $\langle J^\mu \rangle$ along with the transverse variations of $\langle \mathcal{P}_\mu \rangle$ and $\langle T_{\mu\nu} \rangle$. Because the momentum current, spatial stress tensor, and energy current are defined through constrained variations of W , our method is not sufficiently refined to directly compute the longitudinal variations of $\langle \mathcal{P}_\mu \rangle$ or $\langle T_{\mu\nu} \rangle$, nor the variations of the energy current. Rather, we obtain the variation energy current at the end of Subsection 5.5 using the Milne-invariance of the Ward identities.

To proceed we exploit the Milne-invariance of W ,

$$W[n_\mu, h^{\mu\nu}, v^\mu, A_\mu] = W' = W[n_\mu, h^{\mu\nu}, (v')^\mu, (A')_\mu], \quad (5.18)$$

which implies that

$$\begin{aligned}\delta W &= \int d^d x \sqrt{\gamma} \left\{ \delta A_\mu \langle J^\mu \rangle - \delta \bar{v}^\mu \langle \mathcal{P}_\mu \rangle - \delta n_\mu \langle \mathcal{E}^\mu \rangle - \frac{\delta \bar{h}^{\mu\nu}}{2} \langle T_{\mu\nu} \rangle \right\} \\ &= \int d^d x \sqrt{\gamma} \left\{ \delta (A')_\mu \langle J^\mu \rangle' - \delta (\bar{v}')^\mu \langle \mathcal{P}_\mu \rangle' - \delta n_\mu \langle \mathcal{E}^\mu \rangle' - \frac{\delta \bar{h}^{\mu\nu}}{2} \langle T_{\mu\nu} \rangle' \right\}.\end{aligned}\quad (5.19)$$

Using $(v')^\mu = v^\mu + h^{\mu\nu} \psi_\nu$ and $(A')_\mu = A_\mu + P_\mu^\nu \psi_\nu - \frac{1}{2} n_\mu \psi^2$, we find

$$\begin{aligned}\delta (\bar{v}')^\mu &= \delta \bar{v}^\mu + \delta \bar{h}^{\mu\nu} \psi_\nu, \\ \delta (A')_\mu &= \delta A_\mu - n_\mu \left(\delta \bar{v}^\nu \psi_\nu + \frac{1}{2} \delta \bar{h}^{\nu\rho} \psi_\nu \psi_\rho \right),\end{aligned}\quad (5.20)$$

where we have set the variation of n_μ to vanish as we are not computing the Milne variation of the energy current. Substituting into (5.19) we obtain

$$\langle J^\mu \rangle' = \langle J^\mu \rangle, \quad (5.21)$$

and the transverse Milne variations of $\langle \mathcal{P}_\mu \rangle$ and $\langle T_{\mu\nu} \rangle$, i.e.

$$\begin{aligned}\langle \mathcal{P}^\mu \rangle' &= \langle \mathcal{P}^\mu \rangle - h^{\mu\nu} \psi_\nu n_\rho \langle J^\rho \rangle, \\ \langle T^{\mu\nu} \rangle' &= \langle T^{\mu\nu} \rangle - (\langle \mathcal{P}^\mu \rangle h^{\nu\rho} + \langle \mathcal{P}^\nu \rangle h^{\mu\rho}) \psi_\rho + h^{\mu\rho} h^{\nu\sigma} \psi_\rho \psi_\sigma n_\alpha \langle J^\alpha \rangle.\end{aligned}\quad (5.22)$$

Note that the Milne variation of $\langle \mathcal{P}^\mu \rangle$ is exactly what we get from the Milne Ward identity $\langle \mathcal{P}^\mu \rangle = P^\mu_\nu \langle J^\nu \rangle$ upon using that the $U(1)$ current is Milne-invariant. From (5.21) and (5.22) we define a *Milne-invariant stress tensor*

$$\langle \mathcal{T}^{\mu\nu} \rangle = \langle T^{\mu\nu} \rangle + \langle \mathcal{P}^\mu \rangle v^\nu + \langle \mathcal{P}^\nu \rangle v^\mu + v^\mu v^\nu n_\rho \langle J^\rho \rangle, \quad (5.23)$$

which will be rather useful below and in our companion papers.

5.4 Weyl Ward identity

Recall our proposal in Subsection 4.1 for the coupling of scale-invariant, Galilean-invariant field theories to \mathcal{M} , namely to impose invariance under the action of “Weyl” transformations (4.2). The corresponding Ward identity comes from $\delta_\Omega W = 0$, where δ_Ω denotes the action (4.3) of an infinitesimal Weyl transformation. This readily gives the Weyl Ward identity

$$z n_\mu \langle \mathcal{E}^\mu \rangle - h^{\mu\nu} \langle T_{\mu\nu} \rangle = 0. \quad (5.24)$$

5.5 Ward identities, simplified

In obtaining the reparameterization Ward identities in (5.17), we did not use the Milne Ward identity. We presently use it and the Milne-invariant stress tensor (5.23) to dramatically simplify the result. We begin with the right-hand-side of the energy Ward identity. It is useful to decompose the derivative of the velocity using $v^\nu D_\nu v^\mu = -F^\mu_\nu v^\nu$ and

$D^\mu v^\nu - D^\nu v^\mu = F^{\mu\nu}$, which gives

$$D_\mu v^\nu = -n_\mu E^\nu + \frac{1}{2} B_{\mu\nu} + \sigma_\mu{}^\nu + \frac{1}{d-1} P_\mu^\nu \vartheta, \quad (5.25)$$

where

$$\begin{aligned} E_\mu &= F_{\mu\nu} v^\nu, & B_{\mu\nu} &= P_\mu^\rho P_\nu^\sigma F_{\rho\sigma}, \\ \vartheta &= D_\mu v^\mu, & \sigma^{\mu\nu} &= \frac{1}{2} \left(D^\mu v^\nu + D^\nu v^\mu - \frac{2}{d-1} h^{\mu\nu} \vartheta \right). \end{aligned}$$

Using $\langle T_{\mu\nu} \rangle = h_{\mu\rho} h_{\nu\sigma} \langle \mathcal{T}^{\rho\sigma} \rangle$ and $\langle J^\mu \rangle = n_\nu \langle \mathcal{T}^{\mu\nu} \rangle$, we find

$$v^\mu F_{\mu\nu} \langle J^\nu \rangle + \frac{1}{2} (D^\mu v^\nu + D^\nu v^\mu) \langle T_{\mu\nu} \rangle = h_{\rho(\mu} D_{\nu)} v^\rho \langle \mathcal{T}^{\mu\nu} \rangle. \quad (5.26)$$

Next, we compute the divergence of $\langle \mathcal{T}^{\mu\nu} \rangle$,

$$\begin{aligned} (D_\nu - \mathcal{G}_\nu) \langle \mathcal{T}^{\mu\nu} \rangle &= (D_\nu - \mathcal{G}_\nu) \langle T^{\mu\nu} \rangle + D_\nu (v^\nu \langle \mathcal{P}^\mu \rangle) + \langle \mathcal{P}^\nu \rangle D_\nu v^\mu \\ &\quad + v^\mu (D_\nu - \mathcal{G}_\nu) \left[\left(\langle \mathcal{P}^\nu \rangle - P_\rho^\nu \langle J^\rho \rangle \right) + \langle J^\nu \rangle \right] - F^\mu{}_\nu v^\nu n_\rho \langle J^\rho \rangle, \end{aligned} \quad (5.27)$$

where we have also used $v^\nu D_\nu v^\mu = -F^\mu{}_\nu v^\nu$ to obtain the field strength in the second line. Using the $U(1)$ and Milne Ward identities, we eliminate the term proportional to v^μ in the second line. Using $D^\mu v^\nu - D^\nu v^\mu = F^{\mu\nu}$ and Milne identity, this can be further simplified to

$$(D_\nu - \mathcal{G}_\nu) \langle \mathcal{T}^{\mu\nu} \rangle = (D_\nu - \mathcal{G}_\nu) \langle T^{\mu\nu} \rangle - v^\nu D^\mu \langle \mathcal{P}_\nu \rangle + D_\nu (v^\nu \langle \mathcal{P}^\mu \rangle) - F^\mu{}_\nu \langle J^\nu \rangle. \quad (5.28)$$

Putting all of the pieces together, the reparameterization Ward identities in (5.17) simplify to

$$\begin{aligned} (D_\mu - \mathcal{G}_\mu) \langle \mathcal{E}^\mu \rangle &= \mathcal{G}_\mu \langle \mathcal{E}^\mu \rangle - h_{\rho(\mu} D_{\nu)} v^\rho \langle \mathcal{T}^{\mu\nu} \rangle, \\ (D_\nu - \mathcal{G}_\nu) \langle \mathcal{T}^{\mu\nu} \rangle &= -(F^n)^\mu{}_\nu \langle \mathcal{E}^\nu \rangle. \end{aligned} \quad (5.29)$$

Using $n_\nu \langle \mathcal{T}^{\mu\nu} \rangle = \langle J^\mu \rangle$, the $U(1)$ Ward identity is just the longitudinal part of the stress tensor identity,

$$n_\mu (D_\nu - \mathcal{G}_\nu) \langle \mathcal{T}^{\mu\nu} \rangle = (D_\mu - \mathcal{G}_\mu) \langle J^\mu \rangle = 0.$$

With (5.29), it is easy to tie up the remaining loose end from Subsection 5.3 and compute the Milne variation of $\langle \mathcal{E}^\mu \rangle$. Using that $\langle \mathcal{T}^{\mu\nu} \rangle$ is Milne-invariant and the Milne variation of the connection (2.15), the left-hand-side of the stress tensor Ward identity has a Milne variation

$$\begin{aligned} \Delta_\psi [(D_\nu - \mathcal{G}_\nu) \langle \mathcal{T}^{\mu\nu} \rangle] &= (\Delta_\psi \Gamma^\mu{}_{\rho\nu}) \langle \mathcal{T}^{\rho\nu} \rangle \\ &= (F^n)^\mu{}_\nu \left(P_\rho^\sigma \psi_\sigma - \frac{1}{2} n_\rho \psi^2 \right) \langle \mathcal{T}^{\nu\rho} \rangle. \end{aligned} \quad (5.30)$$

Comparing with the right-hand-side of the stress tensor Ward identity, we find

$$\langle \mathcal{E}^\mu \rangle' = \langle \mathcal{E}^\mu \rangle - \left(P_\nu^\rho \psi_\rho - \frac{1}{2} n_\nu \psi^2 \right) \langle \mathcal{T}^{\mu\nu} \rangle. \quad (5.31)$$

5.6 The story at $g_s \neq 0$

So far we have derived Ward identities and Milne variations of the currents in the absence of a magnetic moment coupling g_s . We will now do so for $g_s \neq 0$, where we remind the reader that the Milne variation of A_μ is modified as (2.32)

$$(A')_\mu = A_\mu + P_\mu^\nu \psi_\nu - \frac{1}{2} n_\mu \psi^2 + n_\mu \frac{g_s}{4m} \varepsilon^{\nu\rho\sigma} \partial_\nu (n_\rho P_\sigma^\alpha \psi_\alpha).$$

The $U(1)$ and reparameterization Ward identities in (5.17) did not depend on the Milne variation, and so they are unchanged. But now the Milne variation of W is modified as

$$\begin{aligned} \delta_\psi W &= \int d^3x \sqrt{\gamma} \left\{ \left[P_\mu^\nu \psi_\nu + \frac{g_s}{4m} n_\mu \varepsilon^{\nu\rho\sigma} \partial_\nu (n_\rho P_\sigma^\alpha \psi_\alpha) \right] \langle J^\mu \rangle - h^{\mu\nu} \psi_\mu \langle \mathcal{P}_\nu \rangle \right\} \\ &= \int d^3x \sqrt{\gamma} h^{\mu\nu} \psi_\mu \left\{ h_{\nu\rho} \left[\langle J^\rho \rangle - \frac{g_s}{4m} \varepsilon^{\rho\alpha} \partial_\alpha (n_\sigma \langle J^\sigma \rangle) \right] - \langle \mathcal{P}_\nu \rangle \right\} + (\text{boundary term}), \end{aligned} \quad (5.32)$$

where we have used that $\sqrt{\gamma} \varepsilon^{\alpha\beta\gamma} = \epsilon^{\alpha\beta\gamma}$ is just the epsilon symbol along with $\varepsilon^{\mu\nu} = \epsilon^{\rho\mu\nu} n_\rho$. So the Milne Ward identity becomes

$$\langle \mathcal{P}_\mu \rangle = h_{\mu\nu} \left\{ \langle J^\nu \rangle - \frac{g_s}{4m} \varepsilon^{\nu\rho} \partial_\rho (n_\sigma \langle J^\sigma \rangle) \right\}. \quad (5.33)$$

We deduce the Milne variations of $\langle J^\mu \rangle$, $\langle \mathcal{P}_\mu \rangle$, and $\langle T_{\mu\nu} \rangle$ via (5.19). Setting the variations of n_μ to vanish as we are not computing the Milne variation of $\langle \mathcal{E}^\mu \rangle$, we now have

$$\delta(A')_\mu = \delta A_\mu - n_\mu \left(\delta \bar{v}^\nu \psi_\nu + \frac{1}{2} \delta \bar{h}^{\nu\rho} \psi_\nu \psi_\rho \right) + n_\mu \frac{g_s}{4m} \frac{\delta \bar{h}^{\nu\rho} h_{\nu\rho}}{2} \varepsilon^{\alpha\beta\gamma} \partial_\alpha (n_\beta P_\gamma^\delta \psi_\delta), \quad (5.34)$$

where we have used that $\varepsilon^{\alpha\beta\gamma} = \frac{\epsilon^{\alpha\beta\gamma}}{\sqrt{\gamma}}$ with $\epsilon^{\alpha\beta\gamma}$ the epsilon symbol, the variation of the measure for $\delta n_\mu = 0$ is

$$\frac{\delta \sqrt{\gamma}}{\sqrt{\gamma}} = -\frac{1}{2} \delta \gamma^{\mu\nu} \gamma_{\mu\nu} = -\frac{1}{2} \delta \bar{h}^{\mu\nu} h_{\mu\nu}, \quad (5.35)$$

and no term comes from the derivative by virtue of $\delta P_\gamma^\delta = -\delta \bar{v}^\delta n_\gamma$. The same logic that led to (5.22) now gives the transverse Milne variations of the momentum current and stress tensor, which in turn gives

$$\begin{aligned} \langle \mathcal{P}^\mu \rangle' &= \langle \mathcal{P}^\mu \rangle - h^{\mu\nu} \psi_\nu n_\rho \langle J^\rho \rangle, \\ \langle T^{\mu\nu} \rangle' &= \langle T^{\mu\nu} \rangle - (\langle \mathcal{P}^\mu \rangle h^{\nu\rho} + \langle \mathcal{P}^\nu \rangle h^{\mu\rho}) \psi_\rho + h^{\mu\rho} h^{\nu\sigma} \psi_\rho \psi_\sigma n_\alpha \langle J^\alpha \rangle \\ &\quad + h^{\mu\nu} \frac{g_s}{4m} \varepsilon^{\alpha\beta\gamma} \partial_\alpha (n_\beta P_\gamma^\delta \psi_\delta) n_\rho \langle J^\rho \rangle. \end{aligned} \quad (5.36)$$

This implies that the object $\langle \mathcal{T}^{\mu\nu} \rangle$ we defined in (5.23) is no longer Milne-invariant for $g_s \neq 0$. Its variation is

$$\langle \mathcal{T}^{\mu\nu} \rangle' = \langle \mathcal{T}^{\mu\nu} \rangle + h^{\mu\nu} \frac{g_s}{4m} \varepsilon^{\alpha\beta\gamma} \partial_\alpha \left(n_\beta P_\gamma^\delta \psi_\delta \right) n_\rho \langle J^\rho \rangle. \quad (5.37)$$

Note that the variation of the momentum current in (5.36) is what follows from the Milne Ward identity (5.33) upon using that $\langle J^\mu \rangle$ is Milne-invariant.

Next, we reconsider the reparameterization Ward identities in (5.17), starting with the (non-)conservation of the energy current. In terms of $\langle \mathcal{T}^{\mu\nu} \rangle$, the right-hand-side of the third line of (5.17) (minus the $\mathcal{G}_\mu \langle \mathcal{E}^\mu \rangle$ term) simplifies to the expression in (5.26). We do not see how to simplify that further. So we continue with the stress tensor Ward identity. From (5.27), which we obtained without use the Milne Ward identity, we now find

$$\begin{aligned} (D_\nu - \mathcal{G}_\nu) \langle \mathcal{T}^{\mu\nu} \rangle &= (D_\nu - \mathcal{G}_\nu) \langle T^{\mu\nu} \rangle - v^\nu D^\mu \langle \mathcal{P}_\nu \rangle + D_\nu (v^\nu \langle \mathcal{P}^\mu \rangle) - F^\mu{}_\nu \langle J^\nu \rangle \\ &\quad - \frac{g_s}{4m} v^\mu (D_\nu - \mathcal{G}_\nu) \varepsilon^{\nu\rho} \partial_\rho (n_\sigma \langle J^\sigma \rangle) + \frac{g_s}{4m} F^\mu{}_\nu \varepsilon^{\nu\rho} \partial_\rho (n_\sigma \langle J^\sigma \rangle). \end{aligned} \quad (5.38)$$

Using

$$(D_\nu - \mathcal{G}_\nu) \varepsilon^{\nu\rho} \partial_\rho (n_\sigma \langle J^\sigma \rangle) = -\frac{1}{2} \varepsilon^{\nu\rho\sigma} F_{\nu\rho}^n \partial_\sigma (n_\sigma \langle J^\sigma \rangle), \quad (5.39)$$

the reparameterization Ward identities (5.17) become

$$\begin{aligned} (D_\mu - \mathcal{G}_\mu) \langle \mathcal{E}^\mu \rangle &= \mathcal{G}_\mu \langle \mathcal{E}^\mu \rangle - h_{\rho(\mu} D_{\nu)} v^\rho \langle \mathcal{T}^{\mu\nu} \rangle, \\ (D_\nu - \mathcal{G}_\nu) \langle \mathcal{T}^{\mu\nu} \rangle &= -(F^n)^\mu{}_\nu \langle \mathcal{E}^\nu \rangle + \frac{g_s}{4m} \left\{ \frac{v^\mu}{2} \varepsilon^{\nu\rho\sigma} F_{\nu\rho}^n + F^\mu{}_\nu \varepsilon^{\nu\sigma} \right\} \partial_\sigma (n_\alpha \langle J^\alpha \rangle). \end{aligned} \quad (5.40)$$

The longitudinal part of the stress tensor Ward identity is, upon using (5.33), just the conservation of $\langle J^\mu \rangle$ in disguise.

6 Discussion and outlook

In this work we have sought to answer the question of how to couple Galilean-invariant field theories to a background spacetime \mathcal{M} . Our proposal is that one couples the theory to a Newton-Cartan structure, which is parameterized by the data $(n_\mu, h^{\mu\nu}, v^\mu, A_\mu)$ on \mathcal{M} . Here $n_\nu h^{\mu\nu} = 0$, $v^\mu n_\mu = 1$, A_μ is a $U(1)$ connection, and the covariant derivative is defined through (2.3). In coupling the theory to this data, one should maintain invariance under coordinate reparameterization, $U(1)$ gauge transformations, and the Milne boosts (2.13). This last transformation is a spatial vector's worth of shift symmetries, which imposes the covariant version of Galilean boost-invariance.

This proposal passes several tests. In Subsection 2.4, we recovered the centrally extended Galilean algebra as the isotropy algebra of the flat Newton-Cartan structure on \mathbb{R}^d . Galilean field theories can be covariantly coupled to \mathcal{M} as in (2.25). The infinitesimal form of the reparameterization/ $U(1)$ /Milne symmetry transformations reduces to Son's non-relativistic "general covariance" [4], even with a magnetic moment [5, 6], upon gauge-fixing the Milne boost symmetry. See Subsection 2.7 for details.

A somewhat orthogonal check on our proposal comes from holography. In Section 3 we found that Newton-Cartan structures subject to the Milne symmetry come from the boundary geometry of asymptotically Schrödinger spacetimes. So the field theory duals to quantum gravity on Schrödinger spacetimes (see [4, 8]) naturally couple to Newton-Cartan geometry with a Milne-invariant partition function. Somewhat relatedly, as these field theories are often conformal, we also proposed that scale-invariant Galilean theories coupled to \mathcal{M} are invariant under a “Weyl” transformation (4.2) of the Newton-Cartan data.

With the background fields and symmetries in hand, it is easy to derive Ward identities for the one-point functions of the energy current, stress tensor, &c, as we did in Section 5. For the most part, these agreed with the results recently obtained in [6], and the differences can be traced to the fact that one can form tensorial invariants of the non-relativistic “general covariance” in [6] which are not tensors of the Newton-Cartan geometry. We also used the underlying Milne invariance to compute the Milne variations of one-point functions and to greatly simplify the Ward identities, as in (5.29).

We now conclude with a short list of open questions and obvious directions for future work.

1. Many Galilean-invariant field theories are the $c \rightarrow \infty$ limits of relativistic field theories. How is the $c \rightarrow \infty$ limit related to what we have done here? Does a Newton-Cartan structure automatically appear in that limit, replete with the derivative (2.3) and Milne boosts?
2. There are also holographic questions. In Section 3 we showed that a Newton-Cartan structure with the symmetries above appears in the reduction of Lorentzian $d + 1$ -dimensional manifolds along a null isometry. So the field theory duals to quantum gravity on asymptotically Schrödinger spacetimes naturally couple to Newton-Cartan geometry. In particular, the Milne boosts (2.13) correspond to an ambiguity in the identification of the Newton-Cartan data from the $d + 1$ -dimensional metric. If our proposal is correct, then Milne boosts must act on the boundary geometry of all gravity duals of Galilean-invariant field theory. Recently, it was claimed [22, 23] on symmetry grounds that Horava-Lifshitz gravity [24] on spacetimes with certain asymptotics is holographically dual to some Galilean-invariant field theories. In particular, [22] showed that the boundary geometry is comprised of the various background fields appearing in Son’s non-relativistic “general covariance” and that bulk symmetry transformations with support at the boundary act as Son’s non-relativistic diffeomorphisms on that data. If the claim of [22, 23] is correct, then there must be a whole Newton-Cartan structure on the boundary of these gravitational backgrounds, complete with invariance under Milne boosts. The simplest example of the Horava-Lifshitz holography arises from a null reduction of Einstein gravity on AdS_{d+1} , so in that case there will indeed be a Newton-Cartan structure and Milne invariance. The question is whether the more general Horava-Lifshitz gravities lead to this boundary geometry.
3. Relatedly, there are consistent string theory embeddings of quantum gravity on so-called “Lifshitz” spacetimes (introduced in [25, 26]), dual to non-relativistic field theories without Galilean boost invariance. What is the boundary geometry in this

case?¹¹ In field theory terms, what is the correct geometry to which one should couple a non-relativistic field theory without Galilean boosts? A potential answer to this question was given in [1] (which we reviewed in Subsection 2.9), which amounts to a Newton-Cartan structure $(n_\mu, h^{\mu\nu}, v^\mu)$ where all of this data is covariantly constant. We are very sympathetic to this proposal, and would like to see it verified or ruled out by a holographic analysis.

4. What is the Galilean-invariant version of a spinor on \mathcal{M} ? Perhaps one can define a Galilean spinor through the null reduction we mentioned above, provided that the higher-dimensional Lorentzian manifold is spin.
5. Consider a gapped Galilean theory at zero temperature, coupled to \mathcal{M} such that the Newton-Cartan data varies over length scales parametrically longer than the inverse gap. The low-energy effective action may then be expressed as a local functional in a gradient expansion of the background fields. Recently, there has been a great deal of attention devoted to this gradient expansion for topologically non-trivial phases of matter in two spatial dimensions (a partial and somewhat idiosyncratic list of such work is [1, 2, 5, 6, 29–32] and references therein). There one can form Chern-Simons terms out of the background fields, e.g. $A \wedge dA$, which encode transport phenomena of the edge states on the boundary of a finite slab of such material.

These effective actions must be invariant under the symmetries of the problem. In the Galilean-invariant context, Son’s non-relativistic “general covariance” has been used to parameterize the most general low-energy effective action. So presumably the effective actions appearing in e.g. [29] can be written in a way that is invariant under coordinate reparameterizations, $U(1)$ gauge transformations, and Milne boosts. However, there is a puzzle in that we have yet to find the Milne-invariant version of the topological terms appearing in these works. The Chern-Simons term $A \wedge dA$ illustrates the puzzle nicely. The Milne variation of this term at $g_s = 0$ is $2\Phi \wedge dA + \Phi \wedge d\Phi$ where

$$\Phi = \left(P_\mu^\nu \psi_\nu - \frac{1}{2} n_\mu \psi^2 \right) dx^\mu. \quad (6.1)$$

We have yet to find a $U(1)$ and reparameterization-invariant term which can cancel this Milne variation. Similarly, we have yet to see how to redefine the Chern-Simons three-form built out of the gravitational connection (2.3) in a way that is invariant under $U(1)$ gauge transformations and Milne boosts.

At least when $g_s = 0$, it should be possible to construct such a Milne/ $U(1)$ -invariant Chern-Simons term from the null reduction of a Lorentzian manifold in one higher dimension, as we describe in Subsection 3.2. We expect that the Chern-Simons term is encoded in an identically conserved vector built out of the higher-dimensional background.

6. Relatedly, it should be clear that one cannot take various results about Chern-Simons terms and anomalies from relativistic field theory and naïvely apply them in

¹¹Despite the title of [27], we are not aware of an answer to this question in the literature. The authors of [27] study gravity on asymptotically Schrödinger spacetimes in disguise, since their backgrounds can be obtained by reduction from Schrödinger backgrounds, in a similar philosophy to [28].

the Galilean-invariant setting. We further underscore this point below, but for now we explain our concern with an example. There is a folklore theorem (see e.g. [33–36]) in the condensed matter community which implicitly assumes that many features of anomalies in relativistic field theory are present in non-relativistic theories. The claim is that the thermal transport on the boundary of a two-dimensional topologically non-trivial phase is governed by a gravitational anomaly on the edge, signaled in the bulk via a gravitational Chern-Simons term in the low-energy effective action. This chain of logic is fraught with peril. In order to verify it, one must do three things. First, one should obtain the $U(1)$ and Milne-invariant completion of the gravitational Chern-Simons term. Second, one must verify whether the boundary variation of said Chern-Simons term indeed corresponds to an anomaly on the edge. That is, one must see whether that variation may be removed by the addition of a suitable local counterterm on the boundary. Finally, one must use the symmetries of the problem to relate the anomaly to thermal transport. However the only non-perturbative arguments of this sort are those used in [20, 21] for relativistic field theory. Those works crucially employed Riemannian geometry and so do not obviously generalize to the Galilean setting.

7. There are two other questions about Galilean field theory which we tackle in our companion papers [37, 38]. The first is to revisit these theories at nonzero temperature, and the second to initiate a study of anomalies in the context of Newton-Cartan geometry. At nonzero temperature, we recast non-relativistic fluid mechanics in a manifestly reparameterization, $U(1)$, and Milne-invariant way, which we then couple to \mathcal{M} . As a useful example, we determine the first-order hydrodynamics of parity-violating systems in two spatial dimensions, which end up looking rather like the corresponding results [39] for relativistic hydrodynamics in the same setting. We also construct the hydrostatic thermal partition function using the same logic as in relativistic field theory [40–42].

In the second companion paper, we explore two potential classes of anomalies. The first are pure Weyl anomalies for $z = 2$. Exploiting the map in Section 3 between the Newton-Cartan data and a metric on a higher-dimensional manifold with a null isometry, we efficiently solve the Wess-Zumino consistency condition to determine the spectra of potential Weyl anomalies. We do so in detail for theories in two spatial dimensions. We also consider potential flavor and gravitational anomalies. Our approach is selective: we study the anomalous variations that would be natural in a holographic setting, corresponding to Chern-Simons terms in a dual gravitational description on asymptotically Schrödinger spacetimes. However, it turns out that these anomalous variations can be removed by the addition of a suitable local counterterm, which we compute via the transgression machinery of [43]. So these Chern-Simons terms do not correspond to anomalies at all.

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A Details of Newton-Cartan geometry

A.1 The covariant derivative and Milne variations thereof

Here we justify various results quoted in Subsections 2.1 and 2.2. We begin with the covariant derivative D_μ given the Galilei data $(n_\mu, h^{\mu\nu})$. To define D_μ we also introduce v^μ satisfying $n_\mu v^\mu = 1$ and so $h_{\mu\nu}$ via (2.2). We demand that this derivative is compatible with $(n_\mu, h^{\mu\nu})$, that is

$$D_\mu n_\nu = 0, \quad D_\mu h^{\nu\rho} = 0, \quad (\text{A.1})$$

as well as that the spatial part of the torsion $T^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\nu\rho} - \Gamma^\mu{}_{\rho\nu}$ vanishes, i.e. $h_{\mu\sigma} T^\sigma{}_{\nu\rho} = 0$. To determine the constraints this imposes on the connection, we decompose $\Gamma^\mu{}_{\nu\rho}$ into components along and perpendicular to n via

$$\Gamma^\mu{}_{\nu\rho} = v^\mu (\Gamma_v)_{\nu\rho} + h^{\mu\sigma} (\Gamma_h)_{\sigma\nu\rho}. \quad (\text{A.2})$$

Spatial torsionlessness implies that $(\Gamma_h)_{\sigma\nu\rho} = (\Gamma_h)_{\sigma\rho\nu}$. Demanding that n_μ is covariantly constant, we find

$$D_\mu n_\nu = \partial_\mu n_\nu - \Gamma^\rho{}_{\nu\mu} n_\rho = \partial_\mu n_\nu - (\Gamma_v)_{\nu\mu} = 0, \quad (\text{A.3})$$

which immediately gives

$$(\Gamma_v)_{\nu\mu} = \partial_\mu n_\nu. \quad (\text{A.4})$$

This also demonstrates our assertion that we cannot simultaneously maintain the constancy of n_μ and torsionlessness of the derivative when n_μ is not closed.

Covariant constancy of $h^{\mu\nu}$ is then equivalent to

$$h_{\alpha\nu} h_{\beta\rho} D_\mu h^{\nu\rho} = 0, \quad (\text{A.5})$$

as $n_\nu D_\mu h^{\nu\rho} = -h^{\nu\rho} D_\mu n_\nu = 0$. Simplifying, we have

$$\begin{aligned} 0 &= h_{\alpha\nu} h_{\beta\rho} D_\mu h^{\nu\rho} = h_{\alpha\nu} h_{\beta\rho} (\partial_\mu h^{\nu\rho} + \Gamma^\nu{}_{\sigma\mu} h^{\sigma\rho} + \Gamma^\rho{}_{\sigma\mu} h^{\nu\sigma}) \\ &= -P_\alpha^\nu P_\beta^\rho \partial_\mu h_{\nu\rho} + 2P_\alpha^{(\nu} P_\beta^{\rho)} (\Gamma_h)_{\nu\rho\mu}, \end{aligned} \quad (\text{A.6})$$

where we have used

$$\begin{aligned} h_{\alpha\nu} h_{\beta\rho} \partial_\mu h^{\nu\rho} &= h_{\alpha\nu} [\partial_\mu (h_{\beta\rho} h^{\nu\rho}) - h^{\nu\rho} \partial_\mu h_{\beta\rho}] = -h_{\alpha\nu} n_\beta \partial_\mu v^\nu - P_\alpha^\rho (P_\beta^\sigma + v^\sigma n_\beta) \partial_\mu h_{\sigma\rho} \\ &= n_\beta (-h_{\alpha\nu} + P_\alpha^\rho h_{\rho\nu}) \partial_\mu v^\nu - P_\alpha^\nu P_\beta^\rho \partial_\mu h_{\nu\rho} = -P_\alpha^\nu P_\beta^\rho \partial_\mu h_{\nu\rho}. \end{aligned} \quad (\text{A.7})$$

Using that $(\Gamma_h)_{\mu\nu\rho} = (\Gamma_h)_{\mu\rho\nu}$, (A.6) can then be solved to give

$$(\Gamma_h)_{\mu\nu\rho} = \frac{1}{2} (\partial_\nu h_{\mu\rho} + \partial_\rho h_{\mu\nu} - \partial_\mu h_{\nu\rho}) + n_{(\nu} F_{\rho)\mu}, \quad F_{\mu\nu} = -F_{\nu\mu}. \quad (\text{A.8})$$

At this point, F is an arbitrary antisymmetric tensor. Putting the pieces together, the connection is the result we quoted in (2.3),

$$\Gamma^\mu_{\nu\rho} = v^\mu \partial_\rho n_\nu + \frac{1}{2} h^{\mu\sigma} (\partial_\nu h_{\sigma\rho} + \partial_\rho h_{\sigma\nu} - \partial_\sigma h_{\nu\rho}) + h^{\mu\sigma} n_{(\nu} F_{\rho)\sigma}. \quad (\text{A.9})$$

Next, we compute the antisymmetric part of the derivative of v^μ . We have

$$\begin{aligned} 2h_{\rho[\mu} D_{\nu]} v^\rho &= h_{\rho\mu} (\partial_\nu v^\rho + \Gamma^\rho_{\sigma\nu} v^\sigma) - h_{\rho\nu} (\partial_\mu v^\rho + \Gamma^\rho_{\sigma\mu} v^\sigma) \\ &= v^\rho (\partial_\mu h_{\nu\rho} - \partial_\nu h_{\mu\rho}) + v^\sigma \left(P_\mu^\alpha (\Gamma_h)_{\alpha\sigma\nu} - P_\nu^\alpha (\Gamma_h)_{\alpha\sigma\mu} \right) \\ &= 2v^\rho \partial_{[\mu} h_{\nu]\rho} + v^\sigma \left(\partial_\sigma h_{[\mu\nu]} + \partial_{[\nu} h_{\mu]\sigma} - \partial_{[\mu} h_{\nu]\sigma} \right) \\ &\quad - v^\alpha v^\beta \left(\partial_\alpha h_{\beta[\nu} n_{\mu]} + n_{[\mu} \partial_{\nu]} h_{\alpha\beta} - \partial_\alpha h_{\beta[\nu} n_{\mu]} \right) + P_{[\mu}^\alpha F_{\nu]\alpha} + v^\sigma P_{[\mu}^\alpha n_{\nu]} F_{\sigma\alpha} \\ &= -F_{\mu\nu} - v^\alpha \left(n_{[\mu} F_{\nu]\alpha} - F_{\alpha[\mu} n_{\nu]} + n_{[\mu} n_{\nu]} v^\beta F_{\alpha\beta} \right) \\ &= -F_{\mu\nu}. \end{aligned} \quad (\text{A.10})$$

This implies that

$$\begin{aligned} F^\mu_{\nu} v^\nu &= -v^\nu D_\nu v^\rho + v^\mu n_\rho D_\nu v^\rho = -\dot{v}^\mu - v^\mu v^\rho D_\nu n_\rho \\ &= -\dot{v}^\mu, \end{aligned} \quad (\text{A.11})$$

where we have defined the *geodesic acceleration*

$$\dot{v}^\mu \equiv v^\nu D_\nu v^\mu. \quad (\text{A.12})$$

Raising the indices on both sides of (A.10) with $h^{\mu\nu}$, we find that the *curl* of the velocity is

$$D^\mu v^\nu - D^\nu v^\mu = F^{\mu\nu}, \quad (\text{A.13})$$

where $D^\mu = h^{\mu\nu} D_\nu$ and $F^{\mu\nu} = h^{\mu\rho} h^{\nu\sigma} F_{\rho\sigma}$. Putting these together, we see that the necessary and sufficient condition for $F_{\mu\nu}$ to vanish is if v^μ is both *geodesic* and *curl-free*. In this case the Newton-Cartan structure is called a Newton-Cartan-Milne structure [14].

Next we obtain the variation of the connection $\Gamma^\mu_{\nu\rho}$ under Milne boosts. The velocity and $h_{\mu\nu}$ transform under the boost as (2.13a) and (2.13b), and we leave the variation of A_μ arbitrary. Then the variation of $\Gamma^\mu_{\nu\rho}$, which we notate with a Δ_ψ is given by

$$\begin{aligned} \Delta_\psi \Gamma^\mu_{\nu\rho} &= h^{\mu\sigma} \psi_\sigma \partial_\rho n_\nu + \frac{1}{2} h^{\mu\sigma} \left[\partial_\nu \left(-(-n_\rho P_\sigma^\alpha + n_\sigma P_\rho^\alpha) \psi_\alpha + n_\rho n_\sigma \psi^2 \right) \right. \\ &\quad \left. + \partial_\rho \left(-(n_\nu P_\sigma^\alpha + n_\sigma P_\nu^\alpha) \psi_\alpha + n_\nu n_\sigma \psi^2 \right) - \partial_\sigma \left(-(n_\nu P_\rho^\alpha + n_\rho P_\nu^\alpha) \psi_\alpha + n_\nu n_\rho \psi^2 \right) \right] \\ &\quad + h^{\mu\sigma} \left(n_\nu \partial_{[\rho} \Delta_\psi A_{\sigma]} + n_\rho \partial_{[\nu} \Delta_\psi A_{\sigma]} \right) \end{aligned}$$

$$\begin{aligned}
&= h^{\mu\sigma} \left\{ \left(\partial_{[\rho} n_{\nu]} P_{\sigma}^{\alpha} + \partial_{[\sigma} n_{\nu]} P_{\rho}^{\alpha} + \partial_{[\sigma} n_{\rho]} P_{\nu]}^{\alpha} \right) \psi_{\alpha} - \frac{1}{2} \partial_{\sigma} (n_{\nu} n_{\rho} \psi^2) \right. \\
&\quad \left. + n_{\nu} \left(\partial_{[\rho} (\Delta_{\psi} A_{\sigma]} - P_{\sigma]}^{\alpha} \psi_{\alpha}) + \frac{1}{2} \partial_{\rho} n_{\sigma} \psi^2 \right) \right. \\
&\quad \left. + n_{\rho} \left(\partial_{[\nu} (\Delta_{\psi} A_{\sigma]} - P_{\sigma]}^{\alpha} \psi_{\alpha}) + \frac{1}{2} \partial_{\nu} n_{\sigma} \psi^2 \right) \right\} \\
&= h^{\mu\sigma} \left\{ \left(\partial_{[\rho} n_{\nu]} P_{\sigma}^{\alpha} + \partial_{[\sigma} n_{\nu]} P_{\rho}^{\alpha} + \partial_{[\sigma} n_{\rho]} P_{\nu]}^{\alpha} \right) \psi_{\alpha} + \frac{\psi^2}{2} (n_{\nu} \partial_{[\rho} n_{\sigma]} + n_{\rho} \partial_{[\nu} n_{\sigma]}) \right. \\
&\quad \left. + n_{\nu} \partial_{[\rho} (\Delta_{\psi} A_{\sigma]} - P_{\sigma]}^{\alpha} \psi_{\alpha} + \frac{1}{2} n_{\sigma]} \psi^2) + n_{\rho} \partial_{[\nu} (\Delta_{\psi} A_{\sigma]} - P_{\sigma]}^{\alpha} \psi_{\alpha} + \frac{1}{2} n_{\sigma]} \psi^2) \right\}.
\end{aligned} \tag{A.14}$$

The expression in the last equality is the one (2.14) which we quoted in the main text.

A.2 Properties of the curvature tensor

In terms of the connection one-form $\Gamma^{\mu}_{\nu} \equiv \Gamma^{\mu}_{\nu\rho} dx^{\rho}$, the curvature tensor $R^{\mu}_{\nu\rho\sigma}$ is equivalent to the curvature of Γ^{μ}_{ν} ,

$$R^{\mu}_{\nu} \equiv d\Gamma^{\mu}_{\nu} + \Gamma^{\mu}_{\rho} \wedge \Gamma^{\rho}_{\nu} = \frac{1}{2} R^{\mu}_{\nu\rho\sigma} dx^{\rho} \wedge dx^{\sigma}, \tag{A.15}$$

which immediately leads to the Bianchi identity

$$DR^{\mu}_{\nu} = 0. \tag{A.16}$$

Here we have implicitly defined the exterior covariant derivative D which acts on matrix-valued two-forms as $DR^{\mu}_{\nu} = dR^{\mu}_{\nu} + [\Gamma, R]^{\mu}_{\nu}$. The covariant constancy of n_{μ} and $h^{\mu\nu}$ also implies

$$\begin{aligned}
n_{\mu} R^{\mu}_{\nu\rho\sigma} &= -[D_{\rho}, D_{\sigma}] n_{\nu} = 0, \\
h^{\rho(\mu} R^{\nu)}_{\rho\alpha\beta} &= \frac{1}{2} [D_{\alpha}, D_{\beta}] h^{\mu\nu} = 0.
\end{aligned} \tag{A.17}$$

We now turn to the Newtonian condition, which we discussed at the end of Subsection 2.1. After some straightforward and tedious calculation using the definition of the connection (2.3), we find that the Riemann curvature obeys

$$R^{[\mu}_{(\nu}{}^{\rho]}{}_{\sigma)} = \frac{1}{2} h^{\mu\alpha} h^{\nu\beta} n_{(\nu} (dF)_{\sigma)\alpha\beta} + 2(F^n)^{[\mu}_{\alpha} h^{\rho][\gamma} v^{\alpha]} n_{(\nu} h_{\sigma)\beta} D_{\gamma} v^{\beta}, \tag{A.18}$$

where we remind the reader that we have denoted

$$(dF)_{\mu\nu\rho} = \partial_{\mu} F_{\nu\rho} + \partial_{\nu} F_{\rho\mu} + \partial_{\rho} F_{\mu\nu}.$$

So when $dn = F^n = 0$, demanding that the left-hand-side of (A.18) vanishes imposes $dF = 0$. The analogous condition at $dn \neq 0$ is

$$R^{[\mu}_{(\nu}{}^{\rho]}{}_{\sigma)} - 2(F^n)^{[\mu}_{\alpha} h^{\rho][\gamma} v^{\alpha]} n_{(\nu} h_{\sigma)\beta} D_{\gamma} v^{\beta} = 0, \tag{A.19}$$

which we find rather unenlightening.

B A detailed comparison with Brauner, et al

In this Appendix we compare our construction of Newton-Cartan geometry with the proposal for coupling Galilean theories to \mathcal{M} outlined in [17]. We do this in steps. First, we review the basics of the coset construction of nonlinearly realized symmetries, including an alternative approach to Riemannian geometry. We then recap their work, compare it with our own, and find that the two methods give different results. The work of [17] seems more appropriate to describe the effective action of systems with spontaneously broken spacetime symmetries. However, we find in the last Subsection that their approach can be modified so as to give a coset construction of Newton-Cartan geometry, which matches our results in Subsection 2.9.

B.1 Basics of the coset formalism

Consider a theory with a global symmetry group G which is spontaneously broken to a subgroup H . The coset formalism is designed to compute the tensors which may appear in effective actions which are invariant under G using the Goldstone modes of the symmetry breaking. Another way of thinking about it is the following: given a theory which manifestly preserves a symmetry group H embedded in a larger group G , one can add extra degrees of freedom parameterizing a coset G/H so that the full symmetry group is G .

Let us warm up with the case $G = U(1)$, $H = 1$ for a relativistic field theory, as in the abelian Higgs model. The coset construction involves two ingredients: (i.) the Goldstone mode φ which transforms under local $U(1)$ transformations as $\varphi \rightarrow \varphi + \Lambda$, and (ii.) a background gauge field A_μ which couples to the $U(1)$ symmetry current. The Goldstone mode only appears through its derivative via

$$D_\mu \varphi = \partial_\mu \varphi - A_\mu. \quad (\text{B.1})$$

Now consider a field theory whose effective action S_{eff} is a functional of $D_\mu \varphi$ alone, rather than $\partial_\mu \varphi$ or A_μ separately. Integrating over φ enforces the $U(1)$ Ward identity, as

$$D_\mu \langle J^\mu \rangle = D_\mu \left(\frac{1}{\sqrt{-g}} \frac{\delta W}{\delta A_\mu} \right) = \left\langle \frac{1}{\sqrt{-g}} \frac{\delta S_{eff}}{\delta \varphi} \right\rangle = 0. \quad (\text{B.2})$$

The coset construction generalizes the various elements in this basic example. The generalization of φ is a field $y^\alpha(x)$ which parameterizes elements in the coset G/H . A coset is not usually a Lie group in its own right, but its elements can be represented as elements of G . Let T_a be the generators of the Lie algebra of H and B_α the remaining generators of the Lie algebra of G . Also, suppose that G/H is connected. Then elements of G/H may be represented as

$$U \in G/H \quad U = \exp(iy^\alpha(x)B_\alpha), \quad (\text{B.3})$$

Group multiplication endows U with a left G action. For the $G = U(1)$ example, G/H is the Lie group $U(1)$ and its elements can be parameterized as

$$U = \exp(i\varphi). \quad (\text{B.4})$$

The second ingredient is to introduce a connection \mathcal{A}_μ valued in the algebra of G . The final step is to define the *Maurer-Cartan* form, which generalizes $D_\mu\varphi$ above. It is

$$\omega_{MC} = iU^{-1} (d - i\mathcal{A}) U. \quad (\text{B.5})$$

In order to build actions which are invariant under G , one uses the components of the Maurer-Cartan form rather than the connection \mathcal{A} . Under gauge transformations parameterized by $g(x) \in G$, the connection transforms as

$$\mathcal{A} \rightarrow g (\mathcal{A} + id) g^{-1} \quad (\text{B.6})$$

Meanwhile the y^α transform in a non-trivial way which depends on the G action,

$$\exp(i(y')^\alpha B_\alpha) = g \cdot \exp(iy^\alpha B_\alpha). \quad (\text{B.7})$$

To see how this works, consider a case with an arbitrary G which is completely broken. Then the coset is just G and the G action is just left multiplication. Under a gauge transformation, the Maurer-Cartan form is invariant

$$\omega_{MC} \rightarrow iU^{-1}g^{-1} \left(g\mathcal{A}g^{-1} - igdg^{-1} + id \right) gU = iU^{-1} (\mathcal{A} - id) U = \omega_{MC}. \quad (\text{B.8})$$

So a theory whose effective action is a functional of ω_{MC} is indeed invariant under G upon integrating out the coset fields y^α .

B.2 Riemannian geometry from cosets

Following [17], we will now use this formalism to reconstruct (pseudo-)Riemannian geometry. We will start with G being the Poincaré group $Poincare(d)$ and H the Lorentz subgroup $SO(d-1,1)$. Before presenting the analysis, let us recall why this is guaranteed to work. In terms of a vielbein, the tangent space to a point in \mathcal{M} is isomorphic to \mathbb{R}^d equipped with the Minkowski metric η_{AB} . Of course \mathbb{R}^d equipped with η_{AB} can be represented as the coset $Poincare(d)/SO(d-1,1) = G/H$, where η_{AB} is inherited from the invariant tensor η_{AB} of the Poincaré group. So there is a natural $Poincare(d)$ action on $T\mathcal{M}$.

With that disclaimer out of the way, we continue with the Poincaré algebra. It is generated by rotations R^A_B and momenta P_A , and we use η_{AB} to raise and lower indices. The algebra is defined through

$$\begin{aligned} [R^A_B, R^C_D] &= i \left(\eta^{AC} R_{BD} - \delta_D^A R_B^C - \delta_B^C R^A_D + \eta_{BD} R^{AC} \right), \\ [R^A_B, P_C] &= i \left(\delta_C^A P_B - \eta_{BC} P^A \right). \end{aligned} \quad (\text{B.9})$$

Elements of $G/H \approx \mathbb{R}^d$ can be represented as

$$U = \exp \left(i y^A P_A \right). \quad (\text{B.10})$$

Unlike in the usual setting, the y^A will *not* be dynamical fields. They will instead serve as a means to building a vielbein.

We parameterize the connection \mathcal{A} as

$$\mathcal{A} = p^A P_A + \frac{1}{2} \omega^A{}_B R^B{}_A, \quad (\text{B.11})$$

where $\omega^A{}_B$ is a connection valued in the algebra of $SO(d-1,1)$ and so satisfies $\omega^{AB} = -\omega^{BA}$. We readily find that the Maurer-Cartan form is

$$\omega_{MC} = \left(p^A - dy^A - \omega^A{}_B y^B \right) P_A + \frac{1}{2} \omega^A{}_B R^B{}_A = e^A P_A + \frac{1}{2} \omega^A{}_B R^B{}_A \quad (\text{B.12})$$

where in the second equality we implicitly (and suggestively) define a vector-valued one-form e^A through the expression in parenthesis. Under an infinitesimal gauge transformation

$$g = \exp \left(i \left[\lambda^A P_A + \frac{1}{2} v^A{}_B R^B{}_A \right] \right), \quad (\text{B.13})$$

with $v^{AB} = -v^{BA}$, the y^A and components of \mathcal{A} vary as

$$\begin{aligned} \delta_\chi y^A &= \lambda^A - v^A{}_B y^B, \\ \delta_\chi p^A &= d\lambda^A - v^A{}_B f^B + \omega^A{}_B \lambda^B, \\ \delta_\chi \omega^A{}_B &= dv^A{}_B + \omega^A{}_C v^C{}_B - v^A{}_C \omega^C{}_B. \end{aligned} \quad (\text{B.14})$$

The last line gives the transformation rule for the spin connection, and substituting these variations into the definition of e^A in (B.12) gives the variation of e^A ,

$$\delta_\chi e^A = -v^A{}_B e^B. \quad (\text{B.15})$$

Two observations are in order. First, the elements of the Maurer-Cartan form are invariant under local translations λ^A . Second, in (B.14) and (B.15) we recognize the transformation laws of the inverse vielbein e^A_μ and spin connection under local Lorentz rotations. So the coset formalism in this instance gives us a vielbein and a spin connection and so the basic building blocks of Riemannian geometry, provided that the y^A are non-dynamical. When the y^A are dynamical, this formalism still gives a vielbein and spin connection, but in a way that is ready-made to address aspects of spontaneous symmetry breaking.

B.3 The comparison

In the same spirit let us now take G to be the centrally extended Galilean group. The algebra is spanned by the generators of spatial rotations $R^i{}_j$ with $R^{ij} = -R^{ji}$, Galilean boosts K_i , time translation H , spatial momenta P_i , and particle number M . We use δ^{ij} to

raise and lower spatial indices. Expressed in terms of Hermitian generators, the algebra is

$$\begin{aligned} [R^i_j, R^k_l] &= i \left(\delta^{ik} R_{jl} - \delta^i_l R_j^k - \delta^k_j R_l^i + \delta_{jl} R^{ik} \right), \\ [R^i_j, P_k] &= i \left(\delta^i_k P_j - \delta_{jk} P^i \right), & [R^i_j, K_k] &= i \left(\delta^i_k K_j - \delta_{jk} K^i \right), \\ [P_i, K_j] &= -i \delta_{ij} M, & [H, K_i] &= -i P_i, \end{aligned}$$

with all other commutators vanishing. In what follows we collectively denote H and P_i as P_A , with $P_0 = H$.

The authors of [17] proceed by taking H to be the subgroup $SO(d-1) \times U(1)$ generated by R^i_j and M . Then the coset G/H is not a subgroup, as the remaining generators (K_i, P_A) do not form a subalgebra. Moreover, the tangent space to \mathcal{M} has nothing to do with G/H . Nevertheless the authors of [17] forge ahead by parameterizing G/H through elements of the form

$$U = \exp \left(i y^A P_A \right) \exp \left(i u^i K_i \right), \quad (\text{B.16})$$

where as above the y^A are non-dynamical. However and crucially, the u^i are *dynamical*.

Parameterizing the connection \mathcal{A} as

$$\mathcal{A} = p^A P_A + \omega^i_0 K_i + m M + \frac{1}{2} \omega^i_j R^j_i, \quad (\text{B.17})$$

with $\omega^{ij} = -\omega^{ji}$, the Maurer-Cartan form is

$$\begin{aligned} \omega_{MC} &= (p^0 - dy^0) H + \left(p^i - dy^i - \omega^i_A y^A + u^i (p^0 - dy^0) \right) P_i + \left(\omega^i_0 - du^i - \omega^i_j u^j \right) K_i \\ &\quad + \left(m - \omega^i_0 (y_i + u_i y^0) + u^i (p_i - dy_i) + \frac{p^0 - dy^0}{2} u^2 + \omega^{ij} y_i u_j \right) M + \frac{1}{2} \omega^i_j R^j_i \\ &= f^A P_A + \Omega^i K_i + A M + \frac{1}{2} \omega^i_j R^j_i, \end{aligned} \quad (\text{B.18})$$

where in the last line we have implicitly defined the vector-valued one-form f^A , Ω^i , and A . Under an infinitesimal gauge transformation

$$g = \exp \left(i \left[\lambda^A P_A + v^i_0 K_i + \Lambda M + \frac{1}{2} v^i_j R^j_i \right] \right), \quad (\text{B.19})$$

with $v^{ij} = -v^{ji}$, the y^A , u^i , and components of \mathcal{A} vary as

$$\begin{aligned} \delta_\chi y^A &= \lambda^A - v^A_B y^B, \\ \delta_\chi u^i &= v^i_0 - v^i_j u^j, \\ \delta_\chi p^A &= d\lambda^A - v^A_B p^B + \omega^A_B \lambda^B, \\ \delta_\chi \omega^i_0 &= dv^i_0 + \omega^i_j v^j_0 - v^i_j \omega^j_0, \\ \delta_\chi m &= d\Lambda - v^i_0 p_i + \lambda_i \omega^i_0, \\ \delta_\chi \omega^i_j &= dv^i_j + \omega^i_k v^k_j - v^i_k \omega^k_j. \end{aligned} \quad (\text{B.20})$$

From this we determine the variation of the components of the Maurer-Cartan form

$$\begin{aligned}
\delta_\chi f^0 &= 0, \\
\delta_\chi f^i &= -v^i_A f^A, \\
\delta_\chi \Omega^i &= -v^i_j \Omega^j, \\
\delta_\chi A &= d\left(\Lambda - v^i_0 y_i\right).
\end{aligned} \tag{B.21}$$

Note that the f^A , ω^i_0 , and ω^i_j transform in exactly the same way as the Galilean coframe f^A and spin connection of Newton-Cartan geometry as we described in (2.58). So this construction succeeds in that it gives a Galilean coframe as well as the various connections (ω^i_A, A) of Newton-Cartan geometry. However, it does not match our analysis insofar as there are no Milne boosts.

The other major difference with our analysis is the following. For a local field theory which couples to the coframe f^A and the (ω^i_j, A) components of the connection, the dynamical field u^i only appears algebraically in the action through those fields. Integrating it out yields another local action invariant under the symmetries of the problem. Now consider a local theory which couples to the ω^i_0 components of the connection. In the construction of Brauner et al, those couplings would be introduced through Ω^i , in which u^i appears through derivatives. In this instance integrating out u^i will not lead to a local action. At best one might hope to make the u^i parametrically heavier than the other degrees of freedom in the system, so that the effective description at lower energies is a local Galilean-invariant theory coupled to \mathcal{M} . However it is not clear if the most general Galilean theory may be coupled this way. In particular, it seems unlikely that Schrödinger CFTs can be coupled to \mathcal{M} using this method.

B.4 Newton-Cartan and Milne boosts from cosets

We now present an alternative use of the coset construction that will give Newton-Cartan geometry without introducing additional, dynamical fields. As above, take G to be the Galilean group, but now take H to be the subgroup generated by rotations R^i_j , boosts K_i , and particle number M . (This possibility was raised in [17] but not studied in detail.) Then G/H is a Lie group isomorphic to \mathbb{R}^d , which as a vector space is isomorphic to the tangent space to \mathcal{M} . Moreover, the invariant tensors of the Galilean group descend to invariant tensors δ^{ij} and δ^0_A on the tangent space. These are the invariant tensors of Newton-Cartan geometry, and so this construction is guaranteed to recover the Newton-Cartan structure.

We proceed by parameterizing the elements of G/H by

$$U = \exp\left(iy^A P_A\right). \tag{B.22}$$

As in our coset construction of Riemannian geometry in Appendix B.2, the y^A will be non-dynamical fields which we use to obtain a Galilean frame. Next we parameterize the connection \mathcal{A} as

$$\mathcal{A} = p^A P_A + m M + \omega^i_0 K_i + \frac{1}{2} \omega^i_j R^j_i, \tag{B.23}$$

so that the Maurer-Cartan form is

$$\begin{aligned}\omega_{MC} &= (p^0 - dy^0) H + \left(p^i - dy^i - \omega^i_A y^A\right) P_i + \left(m - \omega^i_0 y_i\right) M + \omega^i_0 K_i + \frac{1}{2} \omega^i_j R^j_i, \\ &= f^A P_A + A M + \omega^i_0 K_i + \frac{1}{2} \omega^i_j R^j_i,\end{aligned}\tag{B.24}$$

where in the last line we have implicitly defined f^A and A . Under an infinitesimal gauge transformation (B.19) the y^A and components of \mathcal{A} transform as

$$\begin{aligned}\delta_\chi y^A &= \lambda^A - v^A_B y^B, \\ \delta_\chi p^A &= d\lambda^A - v^A_B p^B + \omega^A_B \lambda^B, \\ \delta_\chi m &= d\Lambda - v^i_0 p_i + \lambda_i \omega^i_0, \\ \delta_\chi \omega^i_A &= dv^i_A + \omega^i_j v^j_A - v^i_j \omega^j_A.\end{aligned}\tag{B.25}$$

We remind the reader that v^0_A and ω^0_A both vanish. From this we find the gauge variations of the remaining components of the Maurer-Cartan form,

$$\begin{aligned}\delta_\chi f^0 &= 0, \\ \delta_\chi f^i &= -v^i_A f^A, \\ \delta_\chi A &= d\left(\Lambda - v^i_0 y_i\right) - v^i_0 f_i.\end{aligned}\tag{B.26}$$

Note that ω_{MC} is invariant under translations λ^A . Crucially, the f^A and spin connection ω^i_A transform in exactly the same way (2.58) as the Galilei frame and $PGal(d)$ connection in our tangent space construction of Newton-Cartan geometry in Subsection 2.9. So the f^A defined here furnishes a Galilei coframe on \mathcal{M} . Finally, the gauge field transforms under Galilean boosts in the same way (2.63) as we found in our analysis at the end of Subsection 2.9, wherein we fixed the 0-component of the frame as $F^\mu_0 = v^\mu$. To summarize, the data (f^A, ω^i_A, A) obtained here gives the building blocks of Newton-Cartan geometry, provided that we realize Milne boosts through the action of $PGal(d)$ on the tangent space.

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